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Perelman's reduced volume and a gap theorem for the Ricci flow

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Abstract

In this paper, we show that any ancient solution to the Ricci flow with the reduced volume whose asymptotic limit is sufficiently close to that of the Gaussian soliton is isometric to the Euclidean space for all time. This is a generalization of Anderson's result for Ricci-flat manifolds. As a corollary, a gap theorem for gradient shrinking Ricci solitons is also obtained.

1 Introduction

Let us consider a smooth one-parameter family of Riemannian metrics $g(t)$, $t \in [0, T)$ on a manifold M. We call $(M, g(t))$ a *Ricci flow* if it satisfies

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric} \tag{1.1}$$

where Ric denotes the Ricci tensor of $g(t)$. We also use $R := \mathrm{trRic}$ to denote the scalar curvature. The purpose of the present paper is to show a gap theorem for the Ricci flow. In order to state our main theorem, we first recall a heuristic argument given in [25, Section 6].

In his seminal paper [25], Perelman introduced a comparison geometric approach to the Ricci flow, called reduced geometry in [23]. For a Ricci flow $(M^n, g(t)), t \in [0, T)$ with singular time T, take $T_0 < T$ and consider the backward Ricci flow $g(\tau)$, where $\tau := T_0 - t \in [0, T_0]$ is the reverse time. Equipping $\widetilde{M} := M \times S^N \times (0, T_0]$, for large $N \gg 1$, with a metric $\widetilde{g}$ written as

$$\widetilde{g} = g(\tau) + \tau g_{S^N} + \left(R + \frac{N}{2\tau}\right) d\tau^2 \tag{1.2}$$

Perelman observed that $(\widetilde{M}, \widetilde{g})$ has vanishing Ricci curvature up to mod N^{-1}. Here, (S^N, g_{S^N}) is the N-sphere with constant curvature $\frac{1}{2N}$. An easy way

to get a feeling of this is to regard $\widetilde{g}$ as a cone metric by setting $\eta := \sqrt{2N\tau}$. Recall that the metric cone $(N \times (0, T), d\eta^2 + \eta^2 g_N)$ of (N, g_N) is Ricci-flat if and only if $\mathrm{Ric}_{g_N} = (\dim N - 1)g_N$. Then he applied the Bishop-Gromov inequality to $(\widetilde{M}, \widetilde{g})$ formally to obtain an invariant $\tilde{V}_{(p,0)}(\tau)$ which he called the reduced volume. As expected, the reduced volume is non-increasing in τ (Theorem 2.1) and his first application of this was the (re)proof of his no local collapsing theorem [25, Section 7].

Throughout this paper, we adopt the convention that the reduced volume is identically 1 for the Gaussian soliton. The *Gaussian soliton* is the trivial Ricci flow $(\mathbb{R}^n, g_{\mathrm{E}})$ on the Euclidean space regarded as a gradient shrinking Ricci soliton $(\mathbb{R}^n, g_{\mathrm{E}}, \frac{|\cdot|^2}{4})$.

Now we state our main theorem of this paper.

Theorem 1.1. *There exists $\varepsilon_n > 0$ which depends only on $n \geq 2$ and satisfies the following: let $(M^n, g(\tau)), \tau \in [0, \infty)$ be a complete ancient solution to the Ricci flow on an n-manifold M with Ricci curvature bounded below. Suppose that the asymptotic limit of the reduced volume $\lim_{\tau\to\infty} \tilde{V}_{(p,0)}(\tau)$ is greater than $1 - \varepsilon_n$ for some $p \in M$. Then $(M^n, g(\tau))$ is the Gaussian soliton, i.e., isometric to the Euclidean space $(\mathbb{R}^n, g_{\mathrm{E}})$ for all $\tau \in [0, \infty)$.*

We say that $(M, g(\tau))$ is *ancient* when $g(\tau)$ exists for all $\tau \in [0, \infty)$. Ancient solutions are important objects in the study of singularities of the Ricci flow. The limit $\widetilde{\mathcal{V}}(g) := \lim_{\tau\to\infty} \tilde{V}_{(p,0)}(\tau)$ will be called the *asymptotic reduced volume* of the flow $g(\tau)$. We will see in Lemma 3.1 below that $\widetilde{\mathcal{V}}(g)$ is independent of the choice of $p \in M$.

By regarding a Ricci-flat metric as an ancient solution as in Theorem 1.1, we recover the following result, which is the motivation of the present paper.

Theorem 1.2 ([1, Gap Lemma 3.1])**.** *There exists $\varepsilon_n > 0$ which satisfies the following: let (M^n, g) be an n-dimensional complete Ricci-flat Riemannian manifold. Suppose that the asymptotic volume ratio $\nu(g) := \lim_{r\to\infty} \mathrm{Vol}\, B(p, r)/\omega_n r^n$ of g is greater than $1 - \varepsilon_n$. Here ω_n stands for the volume of the unit ball in the Euclidean space $(\mathbb{R}^n, g_{\mathrm{E}})$. Then (M^n, g) is isometric to $(\mathbb{R}^n, g_{\mathrm{E}})$.*

On the way to the proof of Theorem 1.1, we establish several lemmas. Here we state one of them as a theorem, which is of independent interest.

Theorem 1.3. *Let $(M^n, g(\tau)), \tau \in [0, \infty)$ be a complete ancient solution to the Ricci flow on M with Ricci curvature bounded below. If $\widetilde{\mathcal{V}}(g) > 0$,*

then the fundamental group of M is finite. In particular, any ancient κ-solution to the Ricci flow has finite fundamental group.

More generally, Theorem 1.3 is shown for super Ricci flows in Lemma 3.2 under certain assumptions. See also Remark 6.3 below for application.

Finally, we apply the theorems above to gradient shrinkers. We call a triple (M, g, f) a *gradient shrinking Ricci soliton* when

$$\mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{2\lambda} g = 0$$

holds for some positive constant $\lambda > 0$. Shrinking Ricci solitons are typical examples of ancient solutions to the Ricci flow. We normalize the potential function $f \in C^\infty(M)$ by adding a constant so that

$$R + |\nabla f|^2 - \frac{f}{\lambda} = 0 \quad \text{on } M. \tag{1.3}$$

The left-hand side of (1.3) is known to be constant [8, Proposition 1.15].

Corollary 1.1. *Let (M^n, g, f) be a complete gradient shrinking Ricci soliton with Ricci curvature bounded below. Then*

(1) *the fundamental group of M is finite and*

(2) *the normalized f-volume $\int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f}\, d\mu_g$ does not exceed* 1.

(3) *Suppose that*

$$\int_M (4\pi\lambda)^{-n/2} e^{-f} d\mu_g > 1 - \varepsilon_n,$$

then (M^n, g, f) is, up to scaling, the Gaussian soliton $(\mathbb{R}^n, g_{\mathrm{E}}, \frac{|\cdot|^2}{4})$. Here the constant ε_n comes from Theorem 1.1.

Part (1) of Corollary 1.1 is a restatement of the result obtained by many people in more general context (cf. [28]). The other statements in Corollary 1.1 are intimately related to the results of Carrillo–Ni [4]. In particular, Corollary 1.2.(3) proves their conjecture that *the normalized f-volume is* 1 *only for the Gaussian soliton* [4]. See Remark 6.5 below.

The paper is organized as follows. In Section 2, we review definitions and Perelman's results in [25]. We will do this for super Ricci flows. In Section 3, we prove some lemmas required in the proof of the main theorem. In Section 4, we give a proof of Theorem 1.1. In Section 5, we prove Corollary 1.1 and consider expanding solitons with non-negative Ricci curvature. The final section contains some remarks. Appendix A is devoted to detailed proofs of the facts used in the argument without proof.

2 Comparison geometry of super Ricci flows

2.1 Super Ricci flow

In this section, we recall the definitions and results in [25, Sections 6 and 7]. The main references are [25, 29, 17, 8]. Among them, Ye [29] paid careful attention to argue under the assumption of Ricci curvature bounded below rather than bounded sectional curvature (see also [11, Appendix]). The assumption of Theorem 1.1 on the Ricci flow $(M^n, g(\tau))$ is the same as that considered in [29]. We mainly adopt the notation of [8].

We would like to develop Perelman's reduced geometry in more general situation, that is, the super Ricci flow. This will provide us with a convenient setting for comparison geometry of the Ricci flow. A smooth one-parameter family of Riemannian metrics $(M, g(\tau)), \tau \in [0, T)$ is called a *super Ricci flow* when it satisfies

$$\frac{\partial}{\partial \tau} g \leq 2 \operatorname{Ric}. \tag{2.1}$$

Super Ricci flow was introduced by McCann–Topping [20] in their attempt to generalize the contraction property of heat equation in the Wasserstein spaces, which characterizes the non-negativity of the Ricci curvature of the Riemannian metrics (see [26]), to time-depending metrics. See also [19] for this topic.

Basic and important examples of super Ricci flows are

Example 2.1. (1) A solution to the backward Ricci flow equation $\frac{\partial}{\partial \tau} g = 2 \operatorname{Ric}$ and

(2) $g(\tau) := (1 + 2C\tau) g_0, \tau \in [0, \frac{1}{|C| - C})$ for some fixed Riemannian metric g_0 with Ricci curvature bounded from below by $C \in \mathbb{R}$.

Therefore, it can be said that the study of super Ricci flows includes those of (backward) Ricci flows and manifolds with Ricci curvature bounded from below.

We can straightforwardly generalize Perelman's reduced geometry to the super Ricci flow if we impose the following assumptions.

Assumption 2.1. Putting $2h := \frac{\partial}{\partial \tau} g$ and $H := \operatorname{tr}_{g(\tau)} h$, h satisfies

(1) contracted second Bianchi identity $2 \operatorname{div} h(\cdot) = \langle \nabla H, \cdot \rangle$ and

(2) heat-like equation $-\operatorname{tr}_{g(\tau)} \frac{\partial}{\partial \tau} h \geq \Delta_{g(\tau)} H$, or equivalently,

$$-\frac{\partial}{\partial \tau} H \geq \Delta_{g(\tau)} H + 2|h|^2. \tag{2.2}$$

Clearly, the ones in Example 2.1 above satisfy Assumption 2.1. It is known that the evolution equation for the scalar curvature R under the Ricci flow $g(\tau)$ is given by $-\frac{\partial}{\partial\tau}R = \Delta_{g(\tau)}R + 2|\mathrm{Ric}|^2$.

In what follows, we denote by $(M, g(\tau)), \tau \in [0, T)$ a complete super, or backward Ricci flow on an n-manifold M satisfying Assumption 2.1. It is also assumed that the time-derivative $\frac{\partial}{\partial\tau}g$ is bounded from below in each compact time interval, that is, for any compact interval $[\tau_1, \tau_2] \subset [0, T)$, we can find $K = K(\tau_1, \tau_2) \geq 0$ such that $-Kg(\tau) \leq \frac{\partial}{\partial\tau}g \leq 2\,\mathrm{Ric}_{g(\tau)}$ and hence

$$\mathrm{e}^{K(\tau_2-\tau)}g(\tau_2) \geq g(\tau) \geq \mathrm{e}^{-K(\tau-\tau_1)}g(\tau_1)$$

for all $\tau \in [\tau_1, \tau_2]$. Although Assumption 2.1 looks too restrictive, the author's intention is a unified treatment of backward Ricci flows and Riemannian manifolds with non-negative Ricci curvature. (See also Remark 6.1 below.)

2.2 Definition of the reduced volume

Let us start with the definitions. Fix $p \in M$, $[\tau_1, \tau_2] \subset [0, T)$ and $\bar{\tau} \in (0, T)$.

Definition 2.1. Let $\gamma : [\tau_1, \tau_2] \to M$ be a curve. We define the $\mathcal{L}$-*length* of γ and the $\mathcal{L}$-*distance*, respectively, by

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau}\Big(\Big|\frac{d\gamma}{d\tau}\Big|^2_{g(\tau)} + H(\gamma(\tau), \tau)\Big)d\tau$$

and

$$L_{(p,\tau_1)}(q, \tau_2) := \inf\big\{\mathcal{L}(\gamma);\ \gamma : [\tau_1, \tau_2] \to M \text{ with } \gamma(\tau_1) = p, \gamma(\tau_2) = q\big\}.$$

The lower bound of $\frac{\partial}{\partial\tau}g$ guarantees that the $\mathcal{L}$-distance between any two points is achieved by a minimal $\mathcal{L}$-geodesic. This is the only place where we employ the assumption on $\frac{\partial}{\partial\tau}g$. A curve $\gamma(\tau)$ is called an $\mathcal{L}$-*geodesic* when

$$2\nabla_X X + \frac{X}{\tau} - \nabla H + 4h(X, \cdot) = 0, \quad X := \frac{d\gamma}{d\tau}(\tau) \tag{2.3}$$

is satisfied.

Then the *reduced distance* and the *reduced volume* based at $(p, 0)$ are defined, respectively, by

$$\ell_{(p,0)}(q, \bar{\tau}) := \frac{1}{2\sqrt{\bar{\tau}}}L_{(p,0)}(q, \bar{\tau})$$

and

$$\tilde{V}_{(p,0)}(\bar{\tau}) := \int_M (4\pi\bar{\tau})^{-n/2} \mathrm{e}^{-\ell_{(p,0)}(q,\bar{\tau})} d\mu_{g(\bar{\tau})}(q)$$

where $d\mu_{g(\bar{\tau})}$ denotes the volume element induced by $g(\bar{\tau})$.

We can rewrite the reduced volume as

(2.4)
$$\tilde{V}_{(p,0)}(\bar{\tau}) = \int_{T_pM} (4\pi\bar{\tau})^{-n/2} \exp\Big(-\ell_{(p,0)}(\mathcal{L}\ \exp_{\bar{\tau}}(V),\ \bar{\tau})\Big) \mathcal{L}J_V(\bar{\tau}) dx_{g(0)}(V)$$

by pulling back the integrand by the $\mathcal{L}$-*exponential map* $\mathcal{L}\ \exp_{\bar{\tau}} : T_pM \to M$ which assigns $\gamma_V(\bar{\tau})$, if exists, to each $V \in T_pM$. Here γ_V is the $\mathcal{L}$-geodesic determined by $\gamma_V(0) = p$ and $\lim_{\tau\to 0+} \sqrt{\tau}\frac{d\gamma}{d\tau}(\tau) = V$. In (2.4), $dx_{g(0)}$ denotes the Lebesgue measure on T_pM induced by the metric $g(0)$ and $\mathcal{L}J_V(\bar{\tau})$ is called the $\mathcal{L}$-*Jacobian*. Remember that we are using the convention that $\mathcal{L}J_V(\bar{\tau}) = 0$ unless $V \in \Omega_{(p,0)}(\bar{\tau})$. By $V \in \Omega_{(p,0)}(\bar{\tau})$, we mean that $\mathcal{L}\ \exp_{\bar{\tau}}(V)$ exists and lies outside the $\mathcal{L}$-*cut locus* at time $\bar{\tau}$. It follows that $\Omega_{(p,0)}(\bar{\tau})$ is an open set of T_pM, on which $\mathcal{L}\ \exp_{\bar{\tau}}$ is a diffeomorphism, and that $\Omega_{(p,0)}(\tau_2) \subset \Omega_{(p,0)}(\tau_1)$ for $\tau_2 > \tau_1 > 0$. The base point $(p, 0)$ will often be suppressed.

2.3 Monotonicity of the reduced volume

Next, we recall the computations performed in [25, Section 7].

Let $q \in \mathcal{L}\ \exp_{\bar{\tau}}(\Omega_{(p,0)}(\bar{\tau}))$ and $\gamma : [0, \bar{\tau}] \to M$ be the unique minimal $\mathcal{L}$-geodesic from $(p, 0)$ to $(q, \bar{\tau})$. Take a tangent vector $Y \in T_qM$ and extend it to the vector field along γ by solving

$$\nabla_X Y = -h(Y,\cdot) + \frac{Y}{2\tau}, \qquad Y(\bar{\tau}) = Y$$

so that $|Y|^2(\tau) = \frac{\tau}{\bar{\tau}}|Y|^2$.

Then we have that $\nabla\ell(q,\bar{\tau}) = \frac{d\gamma}{d\tau}(\bar{\tau})$ and

$$\frac{\partial}{\partial\tau}\ell(q,\bar{\tau}) = H(q,\bar{\tau}) - \frac{\ell(q,\bar{\tau})}{\bar{\tau}} + \frac{1}{2\bar{\tau}^{3/2}}K$$

$$|\nabla\ell|^2(q,\bar{\tau}) = -H(q,\bar{\tau}) + \frac{\ell(q,\bar{\tau})}{\bar{\tau}} - \frac{1}{\bar{\tau}^{3/2}}K \tag{2.5}$$

$$\mathrm{Hess}\ \ell(Y,Y)(q,\bar{\tau}) \le -h(Y,Y) + \frac{|Y|^2_{g(\bar{\tau})}}{2\bar{\tau}} - \frac{1}{2\sqrt{\bar{\tau}}}\int_0^{\bar{\tau}} \sqrt{\tau}\mathcal{H}(X,Y)\, d\tau$$

$$\Delta\ell(q,\bar{\tau}) \le -H(q,\bar{\tau}) + \frac{n}{2\bar{\tau}} - \frac{1}{2\bar{\tau}^{3/2}}K \tag{2.6}$$

$$\frac{\partial}{\partial\tau}\log\ \mathcal{L}J_V(\bar{\tau}) = \Delta\ell(q,\bar{\tau}) + H(q,\bar{\tau}) \le \frac{n}{2\bar{\tau}} - \frac{1}{2\bar{\tau}^{3/2}}K.$$

Here, following [25, Section 7], we have put

$$\begin{aligned}\mathcal{H}(X) &:= -\frac{\partial H}{\partial \tau} - \frac{H}{\tau} - 2\langle \nabla H, X\rangle + 2h(X,X)\\ K &:= \int_0^{\bar\tau} \tau^{3/2}\mathcal{H}(X)d\tau\\ \mathcal{H}(X,Y) &:= -\langle \nabla_Y \nabla H, Y\rangle + 2\langle R(X,Y)Y, X\rangle + 4\nabla_Y h(X,Y) - 4\nabla_X h(Y,Y)\\ &\quad - 2\frac{\partial h}{\partial \tau}(Y,Y) + 2|h(Y,\cdot)|^2 - \frac{1}{\tau}h(Y,Y).\end{aligned}$$

The point where we have used Assumption 2.1 is the derivation of (2.6) from (2.5) (cf. [8, Lemma 7.42)]):

$$\begin{aligned}&\operatorname{tr}\mathcal{H}(X,\cdot)\\ &\quad = -\Delta H + 2\operatorname{Ric}(X,X) + 4\operatorname{div} h(X) - 4\langle \nabla H, X\rangle - 2\frac{\partial H}{\partial \tau} - 2|h|^2 - \frac{H}{\tau}\\ &\quad = \mathcal{H}(X) + 2\Big[\operatorname{Ric}(X,X) - h(X,X)\Big] + \Big[-\frac{\partial H}{\partial \tau} - \mathit{\Delta} H - 2|h|^2\Big]\\ &\qquad + 2\Big[2\operatorname{div} h(X) - \langle \nabla H, X\rangle\Big]\\ &\quad \geq \mathcal{H}(X).\end{aligned}$$

The quantities corresponding to $\mathcal{H}(X)$ and $\operatorname{tr}\mathcal{H}(X,\cdot)$ appear in [7, (1.2)] and [7, (1.4)] as the trace Harnack expressions of Hamilton [13] and Chow–Hamilton [9], respectively.

We now state the main theorem of this section (cf. [8, 25, 29]).

Theorem 2.1. *Let $(M^n, g(\tau)), \tau \in [0,T)$ be a complete super Ricci flow satisfying Assumption 2.1 with time derivative bounded below. Then for any $p \in M$ and $V \in T_pM$,*

$$(4\pi\tau)^{-n/2}\mathrm{e}^{-\ell_{(p,0)}(\gamma_V(\tau),\tau)}\mathcal{L}J_V(\tau) \tag{2.7}$$

is non-increasing in τ and

$$\lim_{\tau\to 0+}\Big[(4\pi\tau)^{-n/2}e^{-\ell_{(p,0)}(\gamma_V(\tau),\tau)}\mathcal{L}J_V(\tau)\Big] = \pi^{-n/2}e^{-|V|^2_{g(0)}}.$$

Moreover, (2.7) is constant on $(0,\bar\tau]$ if and only if the shrinking soliton equation:

$$\Big[\frac{1}{2}\frac{\partial g}{\partial \tau} + \text{Hess } \ell_{(p,0)} - \frac{1}{2\tau}g\Big](\gamma_V(\tau),\tau) = 0 \tag{2.8}$$

holds along the $\mathcal{L}$-geodesic $\gamma_V(\tau)$ for $\tau \in (0, \bar{\tau}]$.

Hence, $\tilde{V}_{(p,0)}(\tau)$ is non-increasing in τ, $\lim_{\tau \to 0+} \tilde{V}_{(p,0)}(\tau) = 1$ and hence $\tilde{V}_{(p,0)}(\tau) \leq 1$. Moreover, $\tilde{V}_{(p,0)}(\bar{\tau}) = 1$ for some $\bar{\tau} > 0$ if and only if $(M^n, g(\tau)), \tau \in [0, \bar{\tau}]$ is the Gaussian soliton.

We need to give a proof that $\tilde{V}_{(p,0)}(\bar{\tau}) = 1$ for some $\bar{\tau} > 0$ implies that $(M^n, g(\tau))$ is the Gaussian soliton on $[0, \bar{\tau}]$. The proofs of the other statements are minor modifications of those of [8, Lemma 8.16, Corollary 8.17] for the Ricci flow. It should be noted that we have no assumption on the curvature of $g(\tau)$ other than the lower bound of $\frac{\partial}{\partial \tau} g$ in contrast to [8, Corollary 8.17].

Proof of Theorem 2.1. Suppose that $\tilde{V}_{(p,0)}(\bar{\tau}) = 1$. This implies that M is simply connected. Otherwise, the reduced volume of the universal covering $(\bar{M}, \bar{g}(\bar{\tau}))$ of $(M, g(\bar{\tau}))$ must be greater than 1, which is a contradiction.

Fix some small $\tau_\delta \in (0, \bar{\tau})$. For any $\tau \in (\tau_\delta, \bar{\tau}]$, let $\varphi_{\tau - \tau_\delta} : M \to M$ be the map which sends $q \in M$ to $\gamma(\tau)$, where $\gamma : [0, \bar{\tau}] \to M$ is the minimal $\mathcal{L}$-geodesic passing (q, τ_δ) with $\gamma(0) = p$.

Since $\frac{\partial}{\partial \tau} \varphi_{\tau - \tau_\delta}(q) = \frac{d\gamma}{d\tau}(\tau) = \nabla \ell_{(p,0)}(\gamma(\tau), \tau)$, we deduce from (2.8) that

$$\frac{\partial}{\partial \tau} \frac{1}{\tau} (\varphi_{\tau - \tau_\delta})^* g(\tau) = \frac{1}{\tau} (\varphi_{\tau - \tau_\delta})^* \Big[-\frac{1}{\tau} g(\tau) + 2 \operatorname{Hess} \ell_{(p,0)} + \frac{\partial g}{\partial \tau}(\tau) \Big] = 0.$$

Hence,

$$\frac{1}{\tau} (\varphi_{\tau - \tau_\delta})^* g(\tau) = \frac{1}{\tau_\delta} g(\tau_\delta) \text{ or equivalently } g(\tau) = \frac{\tau}{\tau_\delta} (\varphi^{-1}_{\tau - \tau_\delta})^* g(\tau_\delta).$$

Since $g(\tau)$ is smooth around $(p, 0)$, we have

$$\begin{aligned} |\mathrm{Rm}|(q, \tau) &= \frac{\tau_\delta}{\tau} |\mathrm{Rm}|(\varphi^{-1}_{\tau - \tau_\delta}(q), \tau_\delta) \\ &\leq \frac{\tau_\delta}{\tau} \Big(|\mathrm{Rm}|(p, 0) + \theta(\tau_\delta) \Big) \to 0 \text{ as } \tau_\delta \to 0 \end{aligned}$$

where Rm denotes the Riemann curvature tensor and $\theta(\tau_\delta)$ is a function such that $\theta(\tau_\delta) \to 0$ as $\tau_\delta \to 0$. Consequently, $(M^n, g(\tau))$ is flat and hence isometric to $(\mathbb{R}^n, g_{\mathrm{E}})$ for each $\tau \in [0, \bar{\tau}]$. We can write $g(\tau) = u(\tau)^{-1} g_{\mathrm{E}}$ for some positive non-decreasing function $u(\tau)$ with $u(0) = 1$. It remains to show that $u(\tau) = 1$ for all $\tau \in [0, \bar{\tau}]$.

Introduce a new parameter $\sigma := 2\sqrt{\tau}$ to write $g(\sigma) = u(\sigma)^{-1} g_{\mathrm{E}}$ for $\sigma \in [0, \bar{\sigma}]$, where $\bar{\sigma} := 2\sqrt{\bar{\tau}}$. By calculation (cf. [8, Lemma 7.67]), it is easy to see that

$$\ell_{(p,0)}(q, \bar{\tau}) = \frac{d_{\mathrm{E}}(p, q)^2}{\bar{\sigma} \int_0^{\bar{\sigma}} u(\sigma)\, d\sigma} - \frac{n}{2} \log u(\bar{\sigma}) + \frac{n}{2} \frac{\int_0^{\bar{\sigma}} \log u(\sigma)\, d\sigma}{\bar{\sigma}}$$

and

$$\tilde{V}_{(p,0)}(\bar{\tau}) = \int_{\mathbb{R}^n} \Big(\pi\bar{\sigma}^2 \exp \frac{\int_0^{\bar{\sigma}} \log u(\sigma)\, d\sigma}{\bar{\sigma}}\Big)^{-n/2} \exp - \frac{d_{\mathrm{E}}(p,q)^2}{\bar{\sigma}\int_0^{\bar{\sigma}} u(\sigma)\, d\sigma}\, dx(q)$$
$$= \Big(\frac{1}{\bar{\sigma}}\int_0^{\bar{\sigma}} u(\sigma)\, d\sigma\Big)^{n/2}\Big(\exp \frac{\int_0^{\bar{\sigma}} \log u(\sigma)\, d\sigma}{\bar{\sigma}}\Big)^{-n/2}.$$

It follows from Jensen's inequality

$$\frac{1}{\bar{\sigma}}\int_0^{\bar{\sigma}} \log u(\sigma)\, d\sigma \le \log \frac{1}{\bar{\sigma}}\int_0^{\bar{\sigma}} u(\sigma)\, d\sigma \tag{2.9}$$

that $\tilde{V}_{(p,0)}(\bar{\tau}) \ge 1$. As $\tilde{V}_{(p,0)}(\bar{\tau}) \le 1$, we must have equality in (2.9), that is, $u(\bar{\sigma}) = 1$. This completes the proof of Theorem 2.1. □

2.4 Example

As an important example, let us look at a stationary super Ricci flow. Then we obtain an invariant which is called the *static reduced volume* in [8]. Its relation to the volume ratio is given by

Lemma 2.1 ([8, Lemma 8.10])**.** *Let (M^n, g) be an n-dimensional complete Riemannian manifold of non-negative Ricci curvature regarded as a stationary super Ricci flow, i.e., $\frac{\partial}{\partial\tau}g = 0 \le 2\,\mathrm{Ric}$. Then for any $p \in M$ and $\tau > 0$, we have*

$$\tilde{V}_{(p,0)}(\tau) = \int_M (4\pi\tau)^{-n/2} \exp\Big(-\frac{d(p,q)^2}{4\tau}\Big)\, d\mu(q) \le 1, \tag{2.10}$$

and

$$\widetilde{\mathcal{V}}(g) := \lim_{\tau\to\infty} \tilde{V}_{(p,0)}(\tau) = \lim_{r\to\infty} \frac{\mathrm{Vol}\, B(p,r)}{\omega_n r^n} =: \nu(g).$$

Furthermore, the equality holds in (2.10) for some $\tau > 0$ if and only if (M^n, g) is isometric to $(\mathbb{R}^n, g_{\mathrm{E}})$.

By virtue of Lemma 2.1, we know that Theorem 1.1 generalizes Theorem 1.2.

One can easily compute how the reduced distance and reduced volume change under parabolic rescaling.

Proposition 2.1 ([8, Lemma 8.34])**.** *If $g(\tau), \tau \in [0,T)$ is a super Ricci flow, then $(Qg)(\tau) := Qg(Q^{-1}\tau), \tau \in [0, QT)$ is also a super Ricci flow for any $Q > 0$. Under this parabolic rescaling, we have*

$$\ell^{Qg}(q,\tau) = \ell^g(q, Q^{-1}\tau) \text{ and } \tilde{V}^{Qg}(\tau) = \tilde{V}^g(Q^{-1}\tau).$$

In particular, the asymptotic reduced volume is invariant under the parabolic rescaling, i.e., $\widetilde{\mathcal{V}}(g) = \widetilde{\mathcal{V}}(Qg)$, *for any ancient super Ricci flow* $g(\tau), \tau \in [0, \infty)$.

3 Preliminary results

In this section, we prove some lemmas needed in the proof of our main theorem.

3.1 Preliminary estimates

Given a super Ricci flow $(M^n, g(\tau)), \tau \in [0, T)$, take $p \in M$ and $\tau \in (0, T)$. Let us put

$$\mathcal{L}B_\tau(p, r) := \{\mathcal{L}\ \exp_\tau(V);\ V \in \Omega_{(p,0)}(\tau), |V|_{g(0)} < r\}.$$

This notation comes from the fact that a geodesic ball in a Riemannian manifold is the image of ball of the same radius in the tangent space under the exponential map. In this subsection, we derive a few estimates which we shall make heavy use of in the remaining of this paper.

Proposition 3.1. *Let* $u(\cdot, \tau) := (4\pi\tau)^{-n/2} \exp(-\ell_{(p,0)}(\cdot, \tau))$.

(1) For all $r > 0$ *and* $\tau \in (0, T)$, *we have*

$$\tilde{V}_{(p,0)}(\tau) - \varepsilon(r) \leq \int_{\mathcal{L}B_\tau(p,r)} u(\cdot, \tau)\, d\mu_{g(\tau)}.$$

(2) Given $r > 0$ *and* $\tau_0 \in (0, T)$, *we can find a family of subsets* $\mathcal{L}K_{\tau,\tau_0}(p, r)$ *of* M *for* $\tau \in (0, T)$ *satisfying the following properties:*

(a) For all $\tau \leq \tau_0, \mathcal{L}K_{\tau,\tau_0}(p, r)$ *is compact.*

(b) For all $\tau \leq \bar{\tau}, \mathcal{L}K_{\tau,\tau_0}(p, r)$ *contains all of the points* $\gamma(\tau)$ *on any minimal* $\mathcal{L}$*-geodesics* $\gamma : [0, \bar{\tau}] \to M$ *connecting* $(p, 0)$ *and* $(q, \bar{\tau})$ *with* $q \in \mathcal{L}K_{\bar{\tau},\tau_0}(p, r)$.

(c) For all $\tau \geq \tau_0$ *we have*

$$\tilde{V}_{(p,0)}(\tau) - 2\varepsilon(r) \leq \int_{\mathcal{L}K_{\tau,\tau_0}(p,r)} u(\cdot, \tau)\, d\mu_{g(\tau)}.$$

Here, $\varepsilon(r)$ *is a function of* $r > 0$ *with* $\varepsilon(r) \leq \mathrm{e}^{-r^2/2}$ *for all* r *large enough. Clearly,* $\varepsilon(r)$ *decays to* 0 *exponentially as* $r \to \infty$.

Proof. (1) We deduce from (2.4) and Theorem 2.1 that

$$\begin{aligned}\int_{M\setminus\mathcal{L}B_\tau(p,r)} u(\cdot,\tau)\,d\mu_{g(\tau)} &= \int_{\Omega_{(p,0)}(\tau)\setminus B(0,r)} u(\mathcal{L}\exp_\tau(V),\tau)\mathcal{L}J_V(\tau)dx_{g(0)}(V)\\ &\le \int_{T_pM\setminus B(0,r)} \pi^{-n/2}\mathrm{e}^{-|V|^2_{g(0)}}\,dx_{g(0)}(V) =: \varepsilon(r).\end{aligned}$$

(2) Take a compact set K of T_pM so that $K\subset B(0,r)\cap\Omega_{(p,0)}(\tau_0)$ and the Lebesgue measure of $B(0,r)\cap\Omega_{(p,0)}(\tau_0)\setminus K$, induced by $g(0)$, is less than $\pi^{n/2}\varepsilon(r)$. We show that $\mathcal{L}K_{\tau,\tau_0}(p,r) := \mathcal{L}\exp_\tau(K\cap\Omega_{(p,0)}(\tau))$ has the desired properties. It is clear that (a) and (b) hold by construction, since $\Omega_{(p,0)}(\tau_0)\subset\Omega_{(p,0)}(\tau)$ for $\tau\le\tau_0$. Furthermore, by the same argument as in (1), we deduce that

$$\begin{aligned}&\int_{M\setminus\mathcal{L}K_{\tau,\tau_0}(p,r)} u(\cdot,\tau)\,d\mu_{g(\tau)}\\ &\quad= \int_{M\setminus\mathcal{L}B_\tau(p,r)} + \int_{\mathcal{L}B_\tau(p,r)\setminus\mathcal{L}K_{\tau,\tau_0}(p,r)} u(\cdot,\tau)\,d\mu_{g(\tau)} \quad \le 2\varepsilon(r)\end{aligned}$$

for $\tau\ge\tau_0$.

Finally, we estimate $\varepsilon(r)$ for $r\ge r_0$ by

$$\varepsilon(r) = n\omega_n\pi^{-n/2}\int_r^\infty \mathrm{e}^{-r^2}r^{n-1}\,dr \le \int_r^\infty \mathrm{e}^{-r^2/2}r\,dr = \mathrm{e}^{-r^2/2}.$$

Here $r_0\gg 1$ is taken so that $n\omega_n\pi^{-n/2}\mathrm{e}^{-r^2/2}r^{n-2}\le 1$ for all $r\ge r_0$. □

Proposition 3.2. *Assume that $h\ge -C_0g(\tau)$ and $|\nabla H|^2\le D_0$ on $\mathcal{K}\times[0,T_0]$ for some compact set $\mathcal{K}\subset M$ containing a ball $B_{g(0)}(p,r)$. Consider the $\mathcal{L}$-geodesic $\gamma_V:[0,\bar\tau]\to M$ with $\gamma_V(0)=p$ and $\lim_{\tau\to 0+}\sqrt{\tau}\frac{d\gamma_V}{d\tau}=V$. Then we can find $C=C(C_0,T_0), D=D(C_0,D_0,T_0)$ and small $\delta=\delta(C_0,D_0,r,|V|_{g(0)},T_0)>0$ such that*

$$d_{g(0)}(p,\gamma_V(\tau)) \le (C|V|_{g(0)}+D)\sqrt{\tau} \tag{3.1}$$

and hence $\gamma_V(\tau)\in B_{g(0)}(p,r)\subset\mathcal{K}$ for all $\tau\in[0,\delta]$.

Proof. Let $\tau'\in[0,T_0]$ be the maximal time such that $\gamma_V([0,\tau'])\subset\mathcal{K}$. For $\tau\le\tau'$, we use the $\mathcal{L}$-geodesic equation (2.3) to obtain

$$\begin{aligned}\frac{d}{d\tau}|\sqrt{\tau}X|^2_{g(\tau)} &= |X|^2_{g(\tau)} + 2h(\sqrt{\tau}X,\sqrt{\tau}X) + 2\tau\langle\nabla_X X, X\rangle\\ &= -2h(\sqrt{\tau}X,\sqrt{\tau}X) + \tau\langle\nabla H, X\rangle\\ &\le -2h(\sqrt{\tau}X,\sqrt{\tau}X) + |\sqrt{\tau}X|^2_{g(\tau)} + \tau|\nabla H|^2_{g(\tau)}\\ &\le (2C_0+1)|\sqrt{\tau}X|^2_{g(\tau)} + D_0T_0.\end{aligned}$$

From this, we derive that

$$\begin{aligned}|\sqrt{\tau}X|^2_{g(\tau)} &\le \mathrm{e}^{(2C_0+1)\tau}|V|^2_{g(0)} + D_0T_0(\mathrm{e}^{(2C_0+1)\tau}-1)\\ &\le (C|V|_{g(0)}+D)^2\end{aligned}$$

for $C = C(C_0, T_0)$ and $D = D(C_0, D_0, T_0)$, and hence

$$\begin{aligned}d_{g(0)}(p, \gamma_V(\tau)) &\le \int_0^\tau |X|_{g(0)}\, d\tau \le \int_0^\tau \mathrm{e}^{C_0\tau}|X|_{g(\tau)}\, d\tau\\ &\le (C|V|_{g(0)}+D)\int_0^\tau \tau^{-1/2}\, d\tau = (C|V|_{g(0)}+D)\sqrt{\tau}.\end{aligned}$$

As a consequence, we can find $\delta = \delta(C, D, r, |V|_{g(0)}, T_0) > 0$ such that (3.1) holds for $\tau \in [0, \delta]$. □

3.2 Asymptotic reduced volume

Given an ancient super Ricci flow $(M, g(\tau)), \tau \in [0, \infty)$, it is natural to expect that the asymptotic reduced volume $\widetilde{\mathcal{V}}(g) := \lim_{\tau\to\infty} \tilde{V}^g_{(p,0)}(\tau)$ is well defined, namely it does not depend on $p \in M$, as the asymptotic volume ratio is. In this subsection, we prove the following

Lemma 3.1. *Let $(M^n, g(\tau)), \tau \in [0, \infty)$ be a complete ancient super Ricci flow satisfying Assumption 2.1 with time derivative bounded from below. Then for any $(p_k, \tau_k) \in M \times [0, \infty)$ for $k = 1, 2$ with $\tau_2 \ge \tau_1$, we have*

$$\lim_{\tau\to\infty} \tilde{V}^{g_2}_{(p_2,0)}(\tau) \ge \lim_{\tau\to\infty} \tilde{V}^{g_1}_{(p_1,0)}(\tau)$$

where $g_k(\tau) := g(\tau + \tau_k), \tau \in [0, \infty)$. In particular, $\widetilde{\mathcal{V}}(g)$ is well defined.

Proof. Put $\tau_\Delta := \tau_2 - \tau_1 \ge 0$ to notice that $g_2(\tau - \tau_\Delta) = g_1(\tau)$. We first verify

Sublemma 3.1. *For any $(p, \tau_p), (q, \bar{\tau}) \in M \times [0, \infty)$ with $\bar{\tau} > \tau_p \ge \tau_\Delta$,*

$$\frac{1}{2\sqrt{\bar{\tau}-\tau_\Delta}} L^{g_2}_{(p,\tau_p-\tau_\Delta)}(q, \bar{\tau}-\tau_\Delta) \le \frac{1}{2\sqrt{\bar{\tau}}} L^{g_1}_{(p,\tau_p)}(q, \bar{\tau})$$

and

$$\frac{1}{2\sqrt{\bar{\tau}-\tau_\Delta}} L^{g_2}_{(p,\tau_p-\tau_\Delta)}(q, \bar{\tau}-\tau_\Delta) \ge \alpha^{\tau_\Delta}_{\tau_p,\bar{\tau}} \frac{1}{2\sqrt{\bar{\tau}}} L^{g_1}_{(p,\tau_p)}(q, \bar{\tau}),$$

where $\alpha^{\tau_\Delta}_{\tau_p,\bar{\tau}} := \sqrt{\frac{\tau_p-\tau_\Delta}{\tau_p}\frac{\bar{\tau}}{\bar{\tau}-\tau_\Delta}} \ge \sqrt{1-\frac{\tau_\Delta}{\tau_p}}$.

Proof. We use the fact that $H(\cdot,\tau) \geq 0$ for ancient super Ricci flows (Proposition A.1) and the inequality

$$\frac{1}{2\sqrt{\bar\tau}}\sqrt{\tau} \geq \frac{1}{2\sqrt{\bar\tau - \tau_\Delta}}\sqrt{\tau - \tau_\Delta} \quad \text{for all } \tau_p \leq \tau \leq \bar\tau$$

to obtain

$$\begin{aligned}\frac{1}{2\sqrt{\bar\tau}} L^{g_1}_{(p,\tau_p)}(q,\bar\tau) &= \frac{1}{2\sqrt{\bar\tau}} \inf_\gamma \left\{ \int_{\tau_p}^{\bar\tau} \sqrt{\tau}\Big(|\gamma'|^2_{g_1(\tau)} + H_{g_1(\tau)}(\gamma(\tau))\Big)d\tau \right\} \\ &\geq \frac{1}{2\sqrt{\bar\tau - \tau_\Delta}} \inf_\gamma \left\{ \int_{\tau_p}^{\bar\tau} \sqrt{\tau - \tau_\Delta}\Big(|\gamma'|^2_{g_1(\tau)} + H_{g_1(\tau)}(\gamma(\tau))\Big)d\tau \right\} \\ &= \frac{1}{2\sqrt{\bar\tau - \tau_\Delta}} L^{g_2}_{(p,\tau_p - \tau_\Delta)}(q, \bar\tau - \tau_\Delta).\end{aligned}$$

Here inf runs over all curves $\gamma : [\tau_p, \bar\tau] \to M$ with $\gamma(\tau_p) = p$ and $\gamma(\bar\tau) = q$.

To see the second inequality, we use instead

$$\alpha^{\tau_\Delta}_{\tau_p,\bar\tau} \frac{1}{2\sqrt{\bar\tau}}\sqrt{\tau} \leq \frac{1}{2\sqrt{\bar\tau - \tau_\Delta}}\sqrt{\tau - \tau_\Delta} \text{ for all } \tau_p \leq \tau \leq \bar\tau.$$

□

We return to the proof of the lemma. Fix $r > 0$ and $\bar\tau \gg 1$. Take $q \in \mathcal{K}(\bar\tau) := \mathcal{L}^{g_1}K_{\bar\tau, 2\tau_\Delta}(p_1, r)$ and the point $p_\Delta = \gamma(2\tau_\Delta) \in M$ on the minimal $\mathcal{L}^{g_1}$-geodesic $\gamma : [0,\bar\tau] \to M$ from $(p_1, 0)$ to $(q,\bar\tau)$ such that

$$\begin{aligned}L^{g_1}_{(p_1,0)}(q,\bar\tau) &= L^{g_1}_{(p_\Delta,2\tau_\Delta)}(q,\bar\tau) + L^{g_1}_{(p_1,0)}(p_\Delta, 2\tau_\Delta) \\ &\geq L^{g_1}_{(p_\Delta,2\tau_\Delta)}(q,\bar\tau).\end{aligned} \tag{3.2}$$

The inequality in (3.2) is due to the non-negativity of H. Recall that $\mathcal{K} := \mathcal{L}^{g_1}K_{2\tau_\Delta,2\tau_\Delta}(p_1,r)$ is compact and $p_\Delta \in \mathcal{K}$ by construction (Proposition 3.1). It follows from the combination of the triangle inequality for $\mathcal{L}$-distance, Sublemma 3.1 and (3.2) that

$$\begin{aligned}\ell^{g_2}_{(p_2,0)}(q,\bar\tau - \tau_\Delta) &\leq \frac{1}{2\sqrt{\bar\tau - \tau_\Delta}}\Big(L^{g_2}_{(p_\Delta,\tau_\Delta)}(q,\bar\tau - \tau_\Delta) + L^{g_2}_{(p_2,0)}(p_\Delta,\tau_\Delta)\Big) \\ &\leq \frac{1}{2\sqrt{\bar\tau}} L^{g_1}_{(p_\Delta,2\tau_\Delta)}(q,\bar\tau) + \frac{1}{2\sqrt{\bar\tau - \tau_\Delta}} \max_{\mathcal{K}} L^{g_2}_{(p_2,0)}(\cdot,\tau_\Delta) \\ &\leq \ell^{g_1}_{(p_1,0)}(q,\bar\tau) + C(r)\bar\tau^{-1/2}.\end{aligned}$$

Thus, as $\bar\tau > 0$ is large enough,

$$\begin{aligned}
\lim_{\tau\to\infty} \tilde V^{g_2}_{(p_2,0)}(\tau) &\geq \tilde V^{g_2}_{(p_2,0)}(\bar\tau - \tau_\Delta) - \varepsilon(r) \\
&\geq \int_{\mathcal{K}(\bar\tau)} (4\pi\bar\tau)^{-n/2} \exp\Big(-\ell^{g_2}_{(p_2,0)}(\cdot, \bar\tau - \tau_\Delta)\Big) d\mu_{g_2(\bar\tau-\tau_\Delta)} - \varepsilon(r) \\
&\geq \mathrm{e}^{-C(r)\bar\tau^{-1/2}} \int_{\mathcal{K}(\bar\tau)} (4\pi\bar\tau)^{-n/2} \exp\Big(-\ell^{g_1}_{(p_1,0)}(\cdot, \bar\tau)\Big) d\mu_{g_1(\bar\tau)} - \varepsilon(r) \\
&\geq \mathrm{e}^{-C(r)\bar\tau^{-1/2}} \tilde V^{g_1}_{(p_1,0)}(\bar\tau) - 3\varepsilon(r) \\
&\geq \mathrm{e}^{-C(r)\bar\tau^{-1/2}} \lim_{\tau\to\infty} \tilde V^{g_1}_{(p_1,0)}(\tau) - 3\varepsilon(r).
\end{aligned}$$

We have used Proposition 3.1 to derive the fourth inequality. Since $\bar\tau > 0$ and $r > 0$ are arbitrary, the proof of Lemma 3.1 is now complete. $\square$

3.3 Finiteness of fundamental group

Now we are ready to establish Theorem 1.3. As mentioned in the introduction, what we intend to show is the following.

Lemma 3.2. *Let $(M^n, g(\tau)), \tau \in [0,\infty)$ be a complete ancient super Ricci flow satisfying Assumption 2.1 with time derivative bounded below. We lift them to the universal covering $\bar M$ of M to obtain the lifted flow $(\bar M, \bar g(\tau))$. Take $p \in M$ and $\bar p \in \pi^{-1}(p)$, where $\pi : \bar M \to M$ is the projection. Suppose that $\widetilde{\mathcal{V}}(g) := \lim_{\tau\to\infty} \tilde V^{g}_{(p,0)}(\tau) > 0$. Then we have*

$$|\pi_1(M)| = \widetilde{\mathcal{V}}(\bar g)\widetilde{\mathcal{V}}(g)^{-1} < +\infty.$$

Before we begin the proof of Lemma 3.2, let us state the following immediate corollary, which follows from Lemma 3.2 combined with Lemma 2.1.

Corollary 3.1 ([2, 18])**.** *Let (M, g) be a complete Riemmanian manifold with non-negative Ricci curvature and $(\bar M, \bar g)$ be the universal covering of (M, g). If (M, g) has Euclidean volume growth, i.e., $\nu(g) > 0$, then we have*

$$|\pi_1(M)| = \nu(\bar g)\nu(g)^{-1} < +\infty.$$

Here, $\nu(g)$ denotes the asymptotic volume ratio as before.

Proof of Lemma 3.2. The proof is a modification of that of [2, Theorem 1.1]. Fix large $\bar\tau \in (0,\infty)$ and define

$$F := \bigcap_{\alpha\in\pi_1(M)\setminus\{e\}} \Big\{\bar q \in \bar M;\ L^{\bar g}_{(\bar p,0)}(\bar q, \bar\tau) < L^{\bar g}_{(\alpha\bar p,0)}(\bar q, \bar\tau)\Big\}.$$

Then F is a fundamental domain of $\pi : \bar{M} \to M$, namely

$$F \cap \alpha F = \emptyset \quad \text{for } \alpha \in \pi_1(M) \setminus \{e\} \quad \text{and} \quad \bigcup_{\alpha \in \pi_1(M)} \alpha \bar{F} = \bar{M}.$$

We claim that $\pi : \bar{F} \to M$ is locally isometric and surjective. To see this, pick $q \in M$ and connect $(p, 0)$ and $(q, \bar{\tau})$ by a minimal $\mathcal{L}^g$-geodesic $\gamma : [0, \bar{\tau}] \to M$. Then the lift $\bar{\gamma}$ of γ with $\bar{\gamma}(0) = \bar{p}$ is a minimal $\mathcal{L}^{\bar{g}}$-geodesic in $\bar{M}$. Let $\bar{q} := \bar{\gamma}(\bar{\tau})$. Then we have that $\bar{q} \in \bar{F}$ and $\pi(\bar{q}) = q$.

Furthermore, $\bar{F} \setminus F$ has measure 0, since $\pi(\bar{F} \setminus F)$ consists of the points in M such that minimal $\mathcal{L}^g$-geodesic from $(p, 0)$ is not unique. The set of such points has measure 0 [8, Lemma 7.99].

Fix any finite subset $\Gamma \subset \pi_1(M)$ and set $D_\Gamma := \max\{d_{\bar{g}(0)}(\bar{p}, \alpha\bar{p}); \alpha \in \Gamma\}$. Take $C_0 < \infty$ such that $|h| \leq C_0$ on $B_{\bar{g}(0)}(\bar{p}, D_\Gamma + 1) \times [0, 1]$ and $|\nabla H|^2 \leq C_0$ on $B_{g(0)}(p, 1) \times [0, 1]$. Fix $r > 0$. Due to Proposition 3.2, we can find $\delta = \delta(C_0, r) > 0$ such that $d_{\bar{g}(0)}(\bar{\gamma}_V(\tau), \alpha\bar{p}) \leq 1$ for any $\mathcal{L}^{\bar{g}}$-geodesic $\bar{\gamma}_V$ starting from $\alpha\bar{p}$ with $|V|_{\bar{g}(0)} < r$ and $\tau \in [0,\ \delta]$.

For any $\alpha \in \Gamma$ and $\bar{q} \in \mathcal{L}B_{\bar{\tau}}(\alpha\bar{p}, r) \cap \alpha\bar{F}$, let $\bar{\gamma}$ be the minimal $\mathcal{L}^{\bar{g}}$-geodesic from $(\alpha\bar{p}, 0)$ to $(\bar{q}, \bar{\tau})$ in $\bar{M}$ and connect $\bar{p}$ and $\bar{\gamma}(\delta)$ by a minimal $\bar{g}(0)$-geodesic $\xi_{\bar{p}, \bar{\gamma}(\delta)} : [0, \delta] \to \bar{M}$. Define a curve $\hat{\gamma} : [0, \bar{\tau}] \to \bar{M}$ by

$$\hat{\gamma}(\tau) := \begin{cases} \xi_{\bar{p}, \bar{\gamma}(\delta)}(\tau) & \text{on } [0, \delta] \\ \bar{\gamma}(\tau) & \text{on } [\delta, \bar{\tau}] \end{cases}$$

Then, letting $q := \pi(\bar{q})$,

$$\begin{aligned} \ell^{\bar{g}}_{(\bar{p},0)}(\bar{q}, \bar{\tau}) &\leq \frac{1}{2\sqrt{\bar{\tau}}} \mathcal{L}^{\bar{g}}(\hat{\gamma}) \\ &= \frac{1}{2\sqrt{\bar{\tau}}} \Big(\mathcal{L}^{\bar{g}}(\bar{\gamma}) - \mathcal{L}^{\bar{g}}(\bar{\gamma}|_{[0,\delta]}) + \mathcal{L}^{\bar{g}}(\xi_{\bar{p}, \bar{\gamma}(\delta)}) \Big) \\ &\leq \ell^{\bar{g}}_{(\alpha\bar{p},0)}(\bar{q}, \bar{\tau}) + \frac{1}{3\sqrt{\bar{\tau}}} \delta^{3/2} \Big(\mathrm{e}^{2C_0\delta} \Big(\frac{D_\Gamma + 1}{\delta} \Big)^2 + 2nC_0 \Big) \\ &= \ell^g_{(p,0)}(q, \bar{\tau}) + C(\delta, \Gamma) \bar{\tau}^{-1/2} \end{aligned}$$

where we have used that

$$\ell^{\bar{g}}_{(\alpha\bar{p},0)}(\bar{q}, \bar{\tau}) = \ell^g_{(p,0)}(q, \bar{\tau}) \quad \text{for any } \bar{q} \in \mathcal{L}B_{\bar{\tau}}(\alpha\bar{p}, r) \cap \alpha\bar{F}.$$

We apply Proposition 3.1 to obtain that

$$\begin{aligned}\tilde{V}^{\bar{g}}_{(\bar{p},0)}(\bar{\tau}) &\geq \sum_{\alpha\in\Gamma}\int_{\mathcal{L}B_\tau(\alpha\bar{p},r)\cap\alpha\bar{F}}(4\pi\bar{\tau})^{-n/2}\exp\Big(-\ell^{\bar{g}}_{(\bar{p},0)}(\cdot,\bar{\tau})\Big)d\mu_{\bar{g}(\bar{\tau})}\\ &\geq |\Gamma|\int_{\mathcal{L}B_\tau(p,r)}(4\pi\bar{\tau})^{-n/2}\exp\Big(-\ell^{g}_{(p,0)}(\cdot,\bar{\tau})-C(\delta,\Gamma)\bar{\tau}^{-1/2}\Big)d\mu_{g(\bar{\tau})}\\ &\geq \mathrm{e}^{-C(\delta,\Gamma)\bar{\tau}^{-1/2}}|\Gamma|\Big(\tilde{V}^{g}_{(p,0)}(\bar{\tau})-\varepsilon(r)\Big)\end{aligned}$$

and taking $\bar{\tau}\to\infty$ and $r\to\infty$ yields that

$$\widetilde{\mathcal{V}}(\bar{g})\geq|\Gamma|\widetilde{\mathcal{V}}(g)\text{ for any finite subset }\Gamma\subset\pi_1(M). \tag{3.3}$$

Thus, $\pi_1(M)$ is finite and (3.3) holds for $\Gamma=\pi_1(M)$.

On the other hand, since

$$\ell^{\bar{g}}_{(\bar{p},0)}(\bar{q},\tau)\geq\ell^{g}_{(p,0)}(\pi(\bar{q}),\tau)\text{ for any }(\bar{q},\tau)\in\bar{M}\times(0,\infty)$$

we have

$$\tilde{V}^{\bar{g}}_{(\bar{p},0)}(\tau)\leq|\pi_1(M)|\tilde{V}^{g}_{(p,0)}(\tau)$$

and hence $\widetilde{\mathcal{V}}(\bar{g})\leq|\pi_1(M)|\widetilde{\mathcal{V}}(g)$. This finishes the proof of the lemma. □

We close this subsection by giving another corollary of Lemma 3.2.

Corollary 3.2. *Any ancient κ-solution to the Ricci flow has finite fundamental group.*

The proof is immediate since any ancient κ-solution has positive asymptotic reduced volume [8, Lemma 8.38]. Meanwhile, Perelman has shown that *any ancient κ-solution has zero asymptotic volume ratio $\nu(g(\tau))$* [25, Proposition 11.4] (cf. [4]). This is why Corollary 3.2 does not follow from Corollary 3.1, but from Lemma 3.2. See [8, Definition 8.31] for the definition of ancient κ-solution.

3.4 Reduced volume under Cheeger–Gromov convergence

Although we have considered the super Ricci flow so far, Theorem 1.1 is not true for them. From now on, we concentrate on the Ricci flow. To begin with, let us recall Shi's gradient estimate. Shi's derivative estimate was also employed in the proof of the compactness theorem for the Ricci flow [15], which we will use later.

Theorem 3.1 ((Shi's local gradient estimate [14, Theorem 13.1])). *There exists a constant $C(n) < \infty$ satisfying the following: let $(M^n, g(\tau)), \tau \in [0, T_0]$ be a complete backward Ricci flow on an n-manifold M. Assume that the ball $B_{g(T_0)}(p, r)$ is contained in $\mathcal{K}$ and $|\mathrm{Rm}| \le C_0$ on $\mathcal{K} \times [0, T_0]$ for some compact set $\mathcal{K} \subset M$. Then for $\tau \in [0, T_0)$,*

$$|\nabla \mathrm{Rm}|^2(p, \tau) \le C(n) C_0^2 \Big(\frac{1}{r^2} + \frac{1}{T_0 - \tau} + \frac{1}{C_0} \Big). \tag{3.4}$$

Recall that we say that a sequence of pointed backward Ricci flows

$$\{(M_k^n, g_k(\tau), p_k)\}_{k \in \mathbb{Z}^+}, \tau \in [0, T)$$

converges to a backward Ricci flow $(M_\infty^n, g_\infty(\tau), p_\infty), \tau \in [0, T)$ in the C^∞ Cheeger–Gromov sense if there exist open sets U_k of M_∞ with $p_\infty \in U_k$ and $\cup_{k \in \mathbb{Z}^+} U_k = M_\infty$ and deffeomorphisms $\Phi_k : U_k \to V_k := \Phi_k(U_k) \subset M_k$ with $\Phi_k(p_\infty) = p_k$ so that $\{(U_k, \Phi_k^* g_k(\tau))\}_{k \in \mathbb{Z}^+}$ converges to $(M_\infty^n, g_\infty(\tau))$ in the C^∞ topology on each compact set of $M_\infty^n \times [0, T)$.

By carefully investigating the proof of [8, Lemma 7.66], where curvature is assumed to be bounded on the whole of $M_k \times [0, T)$, one can show the following lemma without modification (cf. [8, Lemma 7.66]).

Lemma 3.3. *Let $\{(M_k^n, g_k(\tau), p_k)\}_{k \in \mathbb{Z}^+}, \tau \in [0, T)$ be a converging sequence of pointed backward Ricci flows in the sense of C^∞ Cheeger–Gromov and $(M_\infty^n, g_\infty(\tau), p_\infty), \tau \in [0, T)$ be the limit. Then we have*

$$\limsup_{k \to \infty} \ell^{g_k}_{(p_k, 0)}(\Phi_k(q), \tau) \le \ell^{g_\infty}_{(p_\infty, 0)}(q, \tau) \tag{3.5}$$

for $\tau \in (0, T)$. The equality is achieved in (3.5), with $\limsup$ *replaced by* $\lim$*, provided $(\Phi_k(q), \tau)$ can be joined to $(p_k, 0)$ by a minimal $\mathcal{L}^{g_k}$-geodesic within the image $\Phi_k(\mathcal{K}) \subset M_k$ of some compact set $\mathcal{K} \subset M_\infty$ for all large $k \in \mathbb{Z}^+$.*

Now we verify the convergence of reduced volumes.

Lemma 3.4. *Let $\{(M_k^n, g_k(\tau), p_k)\}_{k \in \mathbb{Z}^+}, \tau \in [0, T)$ be a sequence of pointed backward Ricci flows converging to $(M_\infty^n, g_\infty(\tau), p_\infty)$. Assume that*

$$|\mathrm{Rm}| \le C_0 \text{ on } V_k \times [0, T) \text{ and } \bigcup_{k \in \mathbb{Z}^+} U_k = M_\infty.$$

Then for any $\tau \in (0, T)$,

$$\lim_{k \to \infty} \tilde{V}^{g_k}_{(p_k, 0)}(\tau) = \tilde{V}^{g_\infty}_{(p_\infty, 0)}(\tau). \tag{3.6}$$

Proof. Let us put $u_\star(q,\tau) := (4\pi\tau)^{-n/2}\exp(-\ell^{g_\star}_{(p_\star,0)}(q,\tau))$ for $\star \in \mathbb{Z}^+ \cup \{\infty\}$, and fix $\bar\tau \in (0,T)$ and $T_0 \in (\bar\tau, T)$. Set $V_\infty := M_\infty$.

We invoke Shi's gradient estimate (Theorem 3.1):

$$|\nabla R|^2(\cdot,\tau) \le \frac{C(n)C_0^2}{\min\{C_0, T_0-\tau\}} \text{ on } B_{[0,T_0]}(V_\star, -\sqrt{C_0}) \text{ for } \tau \in [0,T_0)$$

where

$$B_{[0,T_0]}(V_\star, -\sqrt{C_0}) := \{x \in V_\star; B_{g_\star(\tau)}(x, \sqrt{C_0}) \subset V_\star \text{ for all } \tau \in [0,T_0]\}.$$

Fix $r > 0$. Then by Proposition 3.2, we can find $C(r) < \infty$ such that any $\mathcal{L}^{g_\star}$-geodesic $\gamma_V([0,\bar\tau])$ in $M_\star$ with $\gamma_V(0) = p_\star$ and $|V|_{g_\star(0)} \le r$ can not escape from $B_0(p_\star, C(r))$ when $\star$ is sufficiently large or $= \infty$.

Define $\hat u_k(\cdot,\tau) : M_k \to [0,\infty)$ by

$$\hat u_k(q,\tau) := \begin{cases} u_k(q,\tau) & \text{if } q \in \mathcal{L}B_\tau(p_k, r) \\ 0 & \text{otherwise} \end{cases}$$

Then each $\hat u_k(\cdot,\bar\tau)$ has a compact support in $B_{g_k(0)}(p_k, C(r))$ and it follows from Lemma 3.3 that

$$\limsup_{k\to\infty} \hat u_k(\Phi_k(q),\bar\tau) \in \{u_\infty(q,\bar\tau), 0\}. \tag{3.7}$$

Therefore, noting that $\hat u_k(\cdot,\bar\tau) \le (4\pi\bar\tau)^{-n/2}\exp(\frac{1}{3}n(n-1)C_0\bar\tau)$, we derive from Proposition 3.1, Fatou's lemma and (3.7) that

$$\begin{aligned}
\limsup_{k\to\infty} \tilde V^{g_k}_{(p_k,0)}(\bar\tau) - \varepsilon(r) &\le \limsup_{k\to\infty} \int_{\mathcal{L}B_{\bar\tau}(p_k,r)} u_k(\cdot,\bar\tau)\, d\mu_{g_k(\bar\tau)} \\
&= \limsup_{k\to\infty} \int_{B_0(p_\infty,C(r))} \hat u_k(\Phi_k(\cdot),\bar\tau)\, d\mu_{\Phi_k^* g_k(\bar\tau)} \\
&\le \int_{B_0(p_\infty,C(r))} \limsup_{k\to\infty} \hat u_k(\Phi_k(\cdot),\bar\tau)\, d\mu_{\Phi_k^* g_k(\bar\tau)} \\
&\le \tilde V^{g_\infty}_{(p_\infty,0)}(\bar\tau).
\end{aligned}$$

On the other hand, by combining Fatou's lemma and (3.5), we obtain

$$
\begin{aligned}
\liminf_{k\to\infty} \tilde{V}^{g_k}_{(p_k,0)}(\bar{\tau}) &\geq \liminf_{k\to\infty} \int_{\mathcal{L}B_{\bar{\tau}}(p_\infty,r)} u_k(\Phi_k(\cdot),\bar{\tau})\, d\mu_{\Phi_k^* g_k(\bar{\tau})} \\
&\geq \int_{\mathcal{L}B_{\bar{\tau}}(p_\infty,r)} \liminf_{k\to\infty} u_k(\Phi_k(\cdot),\bar{\tau})\, d\mu_{\Phi_k^* g_k(\bar{\tau})} \\
&\geq \int_{\mathcal{L}B_{\bar{\tau}}(p_\infty,r)} u_\infty(\cdot,\bar{\tau})\, d\mu_{g_\infty(\bar{\tau})} \\
&\geq \tilde{V}^{g_\infty}_{(p_\infty,0)}(\bar{\tau}) - \varepsilon(r).
\end{aligned}
$$

We also used Proposition 3.1 to get the last inequality. Since $r > 0$ and $\bar{\tau} \in (0,T)$ are chosen arbitrarily, we conclude that

$$
\lim_{k\to\infty} \tilde{V}^{g_k}_{(p_k,0)}(\tau) = \tilde{V}^{g_\infty}_{(p_\infty,0)}(\tau)
$$

for any $\tau \in (0,T)$. This completes the proof of Lemma 3.4. $\square$

4 Proof of the main theorem

Before proceeding to the proof of Theorem 1.1, we first establish the following technical lemma.

Lemma 4.1. *For any $\alpha > 0$ and $\bar{\tau} > 0$ with $\alpha\bar{\tau}^{-1} > 2$, we can find $\varepsilon_n(\alpha\bar{\tau}^{-1}) > 0$ depending on $\alpha\bar{\tau}^{-1}$ and $n \geq 2$ which satisfies the following: let $(M^n, g(\tau)), \tau \in [0,T), T < \infty$ be a complete backward Ricci flow with Ricci curvature bounded bellow. Put*

$$
M(\alpha) := \big\{(p,s) \in M \times [0,T);\ |\mathrm{Rm}|(p,s) > \alpha(T-s)^{-1}\big\}.
$$

Suppose that the reduced volume based at (p,s) satisfies

$$
\tilde{V}_{(p,s)}(Q^{-1}_{(p,s)}\bar{\tau}) > 1 - \varepsilon_n(\alpha\bar{\tau}^{-1}) \quad \text{at all } (p,s) \in M(\alpha)
$$

with $Q_{(p,s)} := |\mathrm{Rm}|(p,s)$. Here we define $\tilde{V}_{(p,s)}(\bar{\tau})$ as $\tilde{V}^{g_s}_{(p,0)}(\bar{\tau})$ for $g_s(\tau) := g(\tau+s), \tau \in [0,T-s)$. Then $M(\alpha) = \emptyset$, that is,

$$
|\mathrm{Rm}|(\cdot,\tau) \leq \alpha(T-\tau)^{-1} \text{ on } M \times [0,T).
$$

One might notice the similarity of the statement of Lemma 4.1 to those of Perelman's pseudolocality theorem [25, Theorem 10.1] and Ni's ε-regularity theorem [23, Theorem 4.4]. In fact, the proof of Lemma 4.1 follows the same line as those of them. (As the referee report says, there is a close relation between gap and local regularity theorems.)

Proof of Lemma 4.1. We prove by contradiction. Fix $\alpha > 0$ and $\bar{\tau} > 0$ with $\alpha\bar{\tau} > 2$. Assume that we have a sequence $\{(M_k^n, g_k(\tau))\}_{k\in\mathbb{Z}^+}, \tau \in [0, T_k)$ of complete backward Ricci flows with Ricci curvature bounded below such that

- $M_k(\alpha) := \{(p,\tau) \in M_k \times [0, T_k);\ |\mathrm{Rm}|(p,\tau)(T_k - \tau) > \alpha\} \neq \emptyset$ and
- $\tilde{V}^{g_k}_{(p,\tau)}(Q^{-1}_{(p,\tau)}\bar{\tau}) > 1 - k^{-1}$ for any $(p,\tau) \in M_k(\alpha)$, where $Q_{(p,\tau)} := |\mathrm{Rm}|(p,\tau)$.

Applying Perelman's point picking lemma (Lemma A.2) for $(A, B) = (k, \alpha)$, we can find a point $(p_k, \tau_k) \in M_k(\alpha)$ such that $\tilde{V}^{g_k}_{(p_k,\tau_k)}(Q_k^{-1}\bar{\tau}) > 1 - k^{-1}$ and

$$|\mathrm{Rm}|(x,\tau) \leq 2Q_k$$

for $(x,\tau) \in B_{g_k(\tau_k)}(p_k, kQ_k^{-1/2}) \times [\tau_k, \tau_k + \frac{1}{2}Q_k^{-1}\alpha]$, where $Q_k := |\mathrm{Rm}|(p_k, \tau_k)$.

Consider the sequence $\{(M_k^n, \tilde{g}_k(\tau), p_k)\}_{k\in\mathbb{Z}^+}$ of rescaled Ricci flows

$$\tilde{g}_k(\tau) := Q_k g_k(Q_k^{-1}\tau + \tau_k),\ \tau \in [0, \alpha/2].$$

Then every $\tilde{g}_k(\tau)$ has $|\mathrm{Rm}|(p_k, 0) = 1$, $|\mathrm{Rm}| \leq 2$ on $B_{\tilde{g}_k(0)}(p_k, k) \times [0, \alpha/2]$, and $\tilde{V}^{\tilde{g}_k}_{(p_k,0)}(\bar{\tau}) > 1 - k^{-1}$ by Proposition 2.1.

Now we observe that the injectivity radius of $(M_k, \tilde{g}_k(0))$ at p_k is uniformly bounded from below. To see this, we use Proposition 3.2 to get small $\delta = \delta(r) > 0$ so that $\mathcal{L}\exp_\delta(p_k, r) \subset B_{\tilde{g}_k(0)}(p_k, 1)$ for some large $r > 0$ and all large k. Then

$$\begin{aligned} 1 - k^{-1} < \tilde{V}^{\tilde{g}_k}_{(p_k,0)}(\bar{\tau}) &\leq \tilde{V}^{\tilde{g}_k}_{(p_k,0)}(\delta) \\ &\leq (4\pi\delta)^{-n/2} e^{n(n-1)\delta}\mathrm{Vol}_{\tilde{g}_k(0)} B_{\tilde{g}_k(0)}(p_k, 1) + \varepsilon(r) \end{aligned}$$

from which we obtain a uniform lower bound for $\mathrm{Vol}_{\tilde{g}_k(0)} B_{\tilde{g}_k(0)}(p_k, 1)$. The desired lower bound for the injectivity radius follows from Cheeger's lemma.

Since each $(M_k^n, \tilde{g}_k(\tau))$ has a uniform curvature bound and lower bound for the injectivity radius at $(p_k, 0)$, according to Hamilton's compactness theorem [15], we can take a subsequence of $\{(M_k^n, \tilde{g}_k(\tau), p_k)\}_{k\in\mathbb{Z}^+}$ converging to the limit Ricci flow $(M_\infty^n, g_\infty(\tau), p_\infty), \tau \in [0, \alpha/2)$. From Lemma 3.4, we infer that $\tilde{V}^{g_\infty}_{(p_\infty,0)}(\bar{\tau}) = 1$, which implies that the limit $(M_\infty^n, g_\infty(0))$ is isometric to the Euclidean space by Theorem 2.1. This is in conflict with that $|\mathrm{Rm}|(p_\infty, 0) = 1$. The proof of Lemma 4.1 is now complete. □

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1. Take $\varepsilon_n := \varepsilon_n(3) > 0$ from Lemma 4.1. Suppose that $(M^n, g(\tau)), \tau \in [0, \infty)$ is a complete ancient solution to the Ricci flow with Ricci curvature bounded from below satisfying that

$$\widetilde{\mathcal{V}}(g) > 1 - \varepsilon_n.$$

Due to Lemma 3.1 and the monotonicity of the reduced volume, we know that

$$\tilde{V}_{(p,\tau)}(\bar{\tau}) > 1 - \varepsilon_n \text{ for all } (p, \tau) \in M \times [0, \infty) \text{ and } \bar{\tau} > 0.$$

By Lemma 3.2, we know that $\pi_1(M)$ is finite, and applying Lemma 4.1 for all $T > 0$ yields that $(M^n, g(\tau)), \tau \in [0, \infty)$ is flat. The only flat manifold with finite fundamental group is the Euclidean space. Thus $(M^n, g(\tau))$ is isometric to $(\mathbb{R}^n, g_{\mathrm{E}})$ for all $\tau \in [0, \infty)$, i.e., $(M^n, g(\tau)), \tau \in [0, \infty)$ is the Gaussian soliton. This concludes the proof of Theorem 1.1. $\square$

Remark 4.1. Theorem 1.1 may have several variations. (See the questions in [22] for instance.) The following, which also generalizes Theorem 1.2, may be thought of as one of them.

Theorem 4.1. *There exists $\varepsilon'_n > 0$ satisfying the following: let $(M^n, g(\tau))$, $\tau \in [0, \infty)$ be a complete ancient solution to the Ricci flow with bounded non-negative Ricci curvature. Suppose that the asymptotic volume ratio $\nu(g(\tau_0))$ of $g(\tau_0)$ is greater than $1 - \varepsilon'_n$ for some $\tau_0 \in [0, \infty)$. Then $(M^n, g(\tau)), \tau \in [0, \infty)$ is the Gaussian soliton.*

The following proposition was proved by the author by utilizing Cheeger–Colding's volume convergence theorem [5, Theorem 5.9] and Lemma A.1(b).

Proposition 4.1 ([30, Theorem 7])**.** *Let $(M, g(\tau))$ be a complete backward Ricci flow with bounded non-negative Ricci curvature. Then the asymptotic volume ratio $\nu(g(\tau))$ of $g(\tau)$ is constant in τ.*

The proof of Theorem 4.1 is essentially the same as that of Theorem 1.1 and we leave it to the interested reader.

We also comment here that Theorem 4.1 is not true when the ancient solution $g(\tau)$ in the statement is replaced with an *immortal solution* $g(t), t \in [0, \infty)$ to the (forward) Ricci flow. In fact, one can show that any Ricci flow $g(t), t \in [0, T)$ which has bounded non-negative curvature operator and the initial metric $g(0) = g_0$ with positive $\nu(g_0) > 0$ extends to the immortal solution $g(t), t \in [0, \infty)$. (See also the example in [10, Chapter 4, Section 5].)

5 A gap theorem for gradient shrinkers

In this section, we present the proof of Corollary 1.1 and discuss the case of expanding Ricci solitons.

5.1 Shrinking Ricci solitons

We now prove Corollary 1.1. Recall that (M^n, g, f) is a complete gradient shrinking Ricci soliton with Ricci curvature bounded below.

Proof of Corollary 1.1. First, we construct an ancient solution to the Ricci flow. (Recall the proof of Theorem 2.1. See also [10, Theorem 4.1].) Define a one-parameter family of diffeomorphisms $\varphi_\tau : M \to M, \tau \in (0, \infty)$ by

$$\frac{d}{d\tau}\varphi_\tau = \frac{\lambda}{\tau}\nabla f \circ \varphi_\tau \quad \text{and} \quad \varphi_\lambda = \mathrm{id}_M.$$

It is easy to see that the gradient vector field ∇f is complete, thanks to the assumption on the lower bound for Ric. Then we pull back g by $\psi_\tau := \varphi_\tau^{-1}$ so as to obtain a backward Ricci flow $g_0(\tau) := \frac{\tau}{\lambda}(\psi_\tau)^* g, \tau \in (0, \infty)$ with $g_0(\lambda) = g$. Put $g_1(\tau) := g(\tau + 1), \tau \in [0, \infty)$ and fix some point $p \in M$. It suffices to show that

$$\widetilde{\mathcal{V}}(g_1) \geq \int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f}\, d\mu_g \tag{5.1}$$

since the left (resp. right)-hand side of (5.1) is ≤ 1 (resp. >0).

Let us first give a heuristic argument. It seems reasonable to hold that

$$\tilde{V}^{g_0}_{(p,0)}(\tau) = \widetilde{\mathcal{V}}(g_0) = \int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f}\, d\mu_g \qquad \text{for all } \tau > 0$$

(cf. [3]). Then inequality (5.1) will follow from Lemma 3.1, if it is applicable to this case. Of course, the problem arises from the fact that $\tau = 0$ is the singular time for $g_0(\tau)$.

Now we give a rigorous proof. Recall that we have normalized f in (1.3) so that

$$R_{g_0(\tau)} + |\nabla f_\tau|^2_{g_0(\tau)} - \frac{f_\tau}{\tau} = 0 \qquad \text{for } \tau > 0$$

where $f_\tau = f(\cdot, \tau) := (\psi_\tau)^* f = f \circ \psi_\tau$. Since $R_{g_0(\tau)}$ is non-negative (Proposition A.1), so is f_τ. Put $x_1 := \varphi_{\tau_1}(x)$ and $x_2 := \varphi_{\tau_2}(x)$ for some $x \in M$. Then it follows from the argument in [8, p. 344] that $\gamma(\tau) := \varphi_\tau \circ \varphi_{\tau_1}^{-1}(x_1)$ is the $\mathcal{L}^{g_0}$-minimal geodesic from (x_1, τ_1) to (x_2, τ_2) and

$$\frac{1}{2\sqrt{\tau_2}} L^{g_0}_{(x_1,\tau_1)}(x_2, \tau_2) = f(x_2, \tau_1) - \sqrt{\frac{\tau_1}{\tau_2}} f(x_1, \tau_1). \tag{5.2}$$

Fix a compact set $\mathcal{K} \subset M$, $\varepsilon > 0$ and $\bar{\tau} \gg 1$. Take $q \in \varphi_{\bar{\tau}}(\mathcal{K})$ and $p_2 \in \varphi_2(\mathcal{K})$ with $q = \varphi_{\bar{\tau}} \circ \varphi_2^{-1}(p_2)$. From the triangle inequality for $\mathcal{L}$-distance, Sublemma 3.1 and (5.2), it follows that

$$\begin{aligned}
\ell^{g_1}_{(p,0)}(q, \bar{\tau}-1) &\leq \frac{1}{2\sqrt{\bar{\tau}-1}}\Big(L^{g_1}_{(p_2,1)}(q,\bar{\tau}-1) + L^{g_1}_{(p,0)}(p_2,1)\Big) \\
&\leq \frac{1}{2\sqrt{\bar{\tau}}} L^{g_0}_{(p_2,2)}(q,\bar{\tau}) + \frac{1}{2\sqrt{\bar{\tau}-1}} \max_{\varphi_2(\mathcal{K})} L^{g_1}_{(p,0)}(\cdot, 1) \\
&\leq f(q,\bar{\tau}) - \sqrt{\frac{2}{\bar{\tau}}} f(p_2, 2) + C(\mathcal{K})\bar{\tau}^{-1/2} \\
&\leq f(q,\bar{\tau}) + C(\mathcal{K})\bar{\tau}^{-1/2}.
\end{aligned}$$

From this, we deduce that

$$\begin{aligned}
\widetilde{\mathcal{V}}(g_1) &\geq \tilde{V}^{g_1}_{(p,0)}(\bar{\tau}-1) - \varepsilon \\
&\geq \mathrm{e}^{-C(\mathcal{K})\bar{\tau}^{-1/2}} \int_{\varphi_{\bar{\tau}}(\mathcal{K})} (4\pi\bar{\tau})^{-n/2} \mathrm{e}^{-f(q,\bar{\tau})} d\mu_{g_0(\bar{\tau})}(q) - \varepsilon \\
&= \mathrm{e}^{-C(\mathcal{K})\bar{\tau}^{-1/2}} \int_{\mathcal{K}} (4\pi\lambda)^{-n/2} \mathrm{e}^{-f} d\mu_g - \varepsilon.
\end{aligned}$$

We have used the equation

$$\int_M h \circ \psi_{\bar{\tau}} \, d\mu_{(\psi_{\bar{\tau}})^* g} = \int_M h \, d\mu_g \quad \text{for any } h \in L^1(d\mu_g)$$

which follows from the definition of pull back. Inequality (5.1) then follows from the arbitrariness of $\bar{\tau} > 0, \varepsilon > 0$ and $\mathcal{K} \subset M$.

By using (5.1), Theorems 1.1 and 1.3 immediately imply Corollary 1.1. As for (3) of Corollary 1.1, it is easy to see that the Euclidean space regarded as a shrinking soliton is the Gaussian soliton up to scaling (cf. [8, p. 416]). This concludes the proof of Corollary 1.1. $\square$

In the above proof, inequality (5.1) was enough for our purpose, however, we can actually show that the equality holds in (5.1) in the situation of Corollary 1.1. Here we describe the proof of this for future applications.

Proposition 5.1. *Let (M^n, g, f) be a complete gradient shrinking Ricci soliton with Ricci curvature bounded below by $-K \in \mathbb{R}$. Assume that f is normalized so that (1.3) holds. Then, with notation as in the proof of Corollary 1.1, we have*

$$\widetilde{\mathcal{V}}(g_1) = \int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f} \, d\mu_g. \tag{5.3}$$

Proof. Take a sequence $\{\tau_i\}_{i\in\mathbb{Z}^+}$ with $\tau_i \to \infty$ as $i \to \infty$ and put $\alpha_i := \sqrt{1-\frac{1}{\tau_i}}$. Fix $r>0$ and $\bar{\tau}>0$ sufficiently large. For any $q \in \mathcal{K}_i(\bar{\tau}-1) := \mathcal{L}^{g_1}K_{\bar{\tau}-1,\tau_i-1}(p,r)$, take $p_i := \gamma(\tau_i-1) \in \mathcal{K}_i := \mathcal{L}^{g_1}K_{\tau_i-1,\tau_i-1}(p,r)$, where γ is the minimal $\mathcal{L}^{g_1}$-geodesic from $(p,0)$ to $(q,\bar{\tau}-1)$.

It follows from the combination of Sublemma 3.1 and (5.2) that

$$\begin{aligned}
\ell^{g_1}_{(p,0)}(q,\bar{\tau}-1) &= \frac{1}{2\sqrt{\bar{\tau}-1}}\Big(L^{g_1}_{(p_i,\tau_i-1)}(q,\bar{\tau}-1) + L^{g_1}_{(p,0)}(p_i,\tau_i-1)\Big)\\
&\geq \alpha_i\frac{1}{2\sqrt{\bar{\tau}}}L^{g_0}_{(p_i,\tau_i)}(q,\bar{\tau})\\
&= \alpha_i\Big(f(q,\bar{\tau}) - \sqrt{\frac{\tau_i}{\bar{\tau}}}f(p_i,\tau_i)\Big)\\
&\geq \alpha_i\Big(f(q,\bar{\tau}) - \sqrt{\frac{\tau_i}{\bar{\tau}}}\max_{\mathcal{K}_i} f(\cdot,\tau_i)\Big)\\
&= \alpha_i f(q,\bar{\tau}) - C(\tau_i)\bar{\tau}^{-1/2}.
\end{aligned}$$

Recall that $L^{g_1}_{(p,0)}(\cdot,\cdot) \geq 0$, which follows from the non-negativity of the scalar curvature of $g_1(\tau)$ (Proposition A.1), and that $\mathcal{K}_i$ is compact.

Thus, by Proposition 3.1,

$$\begin{aligned}
\widetilde{\mathcal{V}}(g_1) &\leq \Big(1-\frac{1}{\bar{\tau}}\Big)^{n/2}\tilde{V}^{g_1}_{(p,0)}(\bar{\tau}-1) + \varepsilon(r)\\
&\leq \int_{\mathcal{K}_i(\bar{\tau}-1)} (4\pi\bar{\tau})^{-n/2}\exp\Big(-\ell^{g_1}_{(p,0)}(\cdot,\bar{\tau}-1)\Big)d\mu_{g_0(\bar{\tau})} + 3\varepsilon(r)\\
&\leq \mathrm{e}^{C(\tau_i)\bar{\tau}^{-1/2}}\int_{\mathcal{K}_i(\bar{\tau}-1)} (4\pi\bar{\tau})^{-n/2}\mathrm{e}^{-\alpha_i f(\cdot,\bar{\tau})}d\mu_{g_0(\bar{\tau})} + 3\varepsilon(r)\\
&\leq \mathrm{e}^{C(\tau_i)\bar{\tau}^{-1/2}}\int_M (4\pi\lambda)^{-n/2}\mathrm{e}^{-\alpha_i f}d\mu_g + 3\varepsilon(r).
\end{aligned}$$

Now we observe that $\mathrm{e}^{-\alpha_i f}$ is integrable for large $i \in \mathbb{Z}^+$. To do this, let us recall that f-volume $\int_M \mathrm{e}^{-f}d\mu$ is finite if the Bakry–Emery tensor $\mathrm{Ric} + \mathrm{Hess}\, f$ is bounded below by positive constant [21, 27]. In our case, we know that

$$\mathrm{Ric} + \mathrm{Hess}\,\alpha_i f \geq \Big(\frac{\alpha_i}{2\lambda} - (1-\alpha_i)K\Big)g > 0$$

and hence $\int_M \mathrm{e}^{-\alpha_i f}d\mu_g$ makes sense for all large $i \in \mathbb{Z}^+$.

Since $r>0$ and $\bar{\tau}>0$ are arbitrary, we have obtained that

$$\widetilde{\mathcal{V}}(g_1) \leq \int_M (4\pi\lambda)^{-n/2}\mathrm{e}^{-\alpha_i f}d\mu_g \quad (<\infty) \tag{5.4}$$

and the right-hand side of (5.4) converges to the normalized f-volume as $i \to \infty$. Combined with (5.1), this completes the proof of the proposition. $\square$

5.2 Expanding Ricci solitons

Finally, we consider gradient expanders of non-negative Ricci curvature and prove the result corresponding to Corollary 1.1 for them. A *gradient expanding Ricci soliton* is a triple (M, g, f) satisfying

$$\mathrm{Ric} - \mathrm{Hess}\, f + \frac{1}{2\lambda} g = 0$$

for some positive constant $\lambda > 0$. We normalize $f \in C^\infty(M)$ so that $R + |\nabla f|^2 - \lambda^{-1} f = 0$ on M for the expander (M, g, f) too.

Proposition 5.2 ([4])**.** *Let (M^n, g, f) be a complete expanding Ricci soliton with non-negative Ricci curvature. Then*

(1) M^n is diffeomorphic to $\mathbb{R}^n$.

(2) We have

$$\int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f}\, d\mu_g \leq 1 \tag{5.5}$$

and the equality holds if and only if (M^n, g, f) is, up to scaling, the expanding Gaussian soliton $(\mathbb{R}^n, g_{\mathrm{E}}, \frac{|\cdot|^2}{4})$.

We remark that the proposition is a restatement of a result of [4]. Because our proof is simple and purely geometric in contrast to the one in [4], we decided to include it here.

Proof of Proposition 5.2. First, we note that the potential function f is bounded below and $\frac{1}{2\lambda}$-convex, i.e., $\mathrm{Hess}\, f \geq \frac{1}{2\lambda} g > 0$. Therefore, f has the unique critical point $p \in M$ where the minimum value of f is attained. Part (1) of the proposition follows from this.

Next, as in the proof of Corollary 1.1, we construct a self-similar solution to the (forward) Ricci flow $g_0(t) := \frac{t}{\lambda}(\psi_t)^* g, t \in (0, \infty)$ and put $g_1(t) := g_0(t+1), t \in [0, \infty)$.

Define the *forward reduced distance* at $(q, \bar{t}) \in M \times (0, \infty)$ by

$$\ell^+_{(p,0)}(q, \bar{t}) := \frac{1}{2\sqrt{\bar{t}}} \inf\bigl\{\mathcal{L}^+(\gamma);\ \gamma(0) = p, \gamma(\bar{t}) = q\bigr\}$$

where we defined the *forward $\mathcal{L}$-length* $\mathcal{L}^+(\gamma)$ of $\gamma : [0, \bar{t}] \to M$ by

$$\mathcal{L}^+(\gamma) := \int_0^{\bar{t}} \sqrt{t}\Big(\Big| \frac{d\gamma}{dt} \Big|^2_{g_1(t)} + R_{g_1(t)}(\gamma(t)) \Big) dt.$$

Then we consider the formal reduced volume defined by

$$\widehat{V}^{g_1}_{(p,0)}(t) := \int_M (4\pi t)^{-n/2} \mathrm{e}^{-\ell^+_{(p,0)}(\cdot,t)} d\mu_{g_1(t)}. \tag{5.6}$$

We do not care whether $\widehat{V}^{g_1}_{(p,0)}(t)$ is monotone. (This is the case when $g_1(t)$ has bounded non-negative curvature operator or non-negative bi-sectional curvature in the Kähler case [24].)

Since $(M^n, g_1(t))$ has non-negative Ricci curvature, we have

$$\begin{aligned}
\ell^+_{(p,0)}(q, \bar{t}-1) &\geq \frac{1}{2\sqrt{\bar{t}-1}} \inf_{\gamma} \int_0^{\bar{t}-1} \sqrt{t} \Big| \frac{d\gamma}{dt} \Big|^2_{g_1(\bar{t}-1)} dt \\
&= \frac{1}{4(\bar{t}-1)} d_{g_1(\bar{t}-1)}(p,q)^2 \geq \frac{1}{4\lambda} d_g(p, \psi_{\bar{t}}(q))^2
\end{aligned}$$

and by Lemma 2.1,

$$\begin{aligned}
\Big(\frac{\bar{t}-1}{\bar{t}}\Big)^{n/2} \widehat{V}^{g_1}_{(p,0)}(\bar{t}-1) &\leq \int_M (4\pi\bar{t})^{-n/2} \exp\Big(-\frac{1}{4\lambda} d_g(p, \psi_{\bar{t}}(\cdot))^2\Big) d\mu_{g_0(\bar{t})} \\
&= \int_M (4\pi\lambda)^{-n/2} \exp\Big(-\frac{d_g(p,\cdot)^2}{4\lambda}\Big) d\mu_g \\
&\leq 1.
\end{aligned}$$

Then, from the same argument as in the derivation of (5.1) in the proof of Corollary 1.1, we derive that

$$\int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f} d\mu_g \leq \liminf_{t\to\infty} \widehat{V}^{g_1}_{(p,0)}(t)$$

and hence

$$\int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f} d\mu_g \leq \int_M (4\pi\lambda)^{-n/2} \exp\Big(-\frac{d_g(p,\cdot)^2}{4\lambda}\Big) d\mu_g \leq 1 \tag{5.7}$$

which yields (5.5).

When the normalized f-volume is 1, we have equalities in (5.7). Then we know from the equality case of Lemma 2.1 that (M^n, g) is isometric to the Euclidean space. The only way to regard $(\mathbb{R}^n, g_{\mathrm{E}})$ as a gradient expanding Ricci soliton is the Gaussian soliton, up to rescaling. This finishes the proof. $\square$

6 Concluding remarks

In this section, we collect some remarks.

Remark 6.1. Let $(M^n, g(\tau)), \tau \in [0, T)$ be a super Ricci flow $\frac{\partial}{\partial\tau} g =: 2h \leq 2\,\mathrm{Ric}$ satisfying Assumption 2.1 on a closed manifold M. Put $H := \mathrm{tr} h$. Following [25], we define the $\mathcal{W}$*-entropy* for a triple $(g(\tau), f, \tau)$ by

$$\mathcal{W}(g(\tau), f, \tau) = \int_M \Big[\tau(|\nabla f|^2 + H) + f - n\Big] u \, d\mu_{g(\tau)} \tag{6.1}$$

where f is a smooth function on M^n, $\tau > 0$ and $u := (4\pi\tau)^{-n/2}\mathrm{e}^{-f}$.

We evolve u by the conjugate heat equation $\frac{\partial}{\partial\tau} u = \Delta_{g(\tau)} u - Hu$, or equivalently,

$$\frac{\partial f}{\partial \tau} = \Delta_{g(\tau)} f - |\nabla f|^2 + H - \frac{n}{2\tau}.$$

Then, by simple calculation, we obtain the entropy formula for the super Ricci flow:

$$\begin{aligned} &\frac{d}{d\tau}\mathcal{W}(g(\tau), f, \tau) \\ &\quad = -2\tau \int_M \Big[\Big|h + \mathrm{Hess}\, f - \frac{1}{2\tau} g\Big|^2 + (dH - 2\mathrm{div} h)(\nabla f) \\ &\qquad + (\mathrm{Ric} - h)(\nabla f, \nabla f) - \frac{1}{2}\Big(\frac{\partial H}{\partial \tau} + \Delta_{g(\tau)} H + 2|h|^2\Big)\Big] u \, d\mu_{g(\tau)} \\ &\quad \leq 0 \end{aligned}$$

from which we simultaneously recover the entropy formulae of Perelman ($h = \mathrm{Ric}$) [25] and Ni ($h = 0$) [22].

We also have similar formula for the super Ricci flow analogue of $\mathcal{F}$*-entropy* introduced in [25, Section 1].

Remark 6.2. (1) We can find the optimal value ε_n of the constant obtained in Theorem 1.1, namely $\varepsilon_n := 1 - \max\{\widetilde{\mathcal{V}}(g)\} > 0$. We take the maximum over all the complete n-dimensional non-Gaussian ancient solutions to the Ricci flow with Ricci curvature bounded below. The maximum is achieved, as is seen by the limit argument used in the proof of Lemma 4.1. Then it is easy to see that $\{\varepsilon_n\}_{n=2}^{\infty}$ is a non-increasing sequence. It seems interesting to determine the exact value of $\lim_{n\to\infty} \varepsilon_n$.

(2) Now we calculate an asymptotic reduced volume (or normalized f-volume) for the round n-sphere (S^n, g_{S^n}) with constant Ricci curvature $\mathrm{Ric} = \frac{1}{2} g_{S^n}$. Then $g(\tau) := (1+\tau) g_{S^n}, \tau \in [0, \infty)$ is an ancient solution to

the Ricci flow, while (S^n, g_{S^n}, f) with $f \equiv \frac{n}{2}$ is a gradient shrinking Ricci soliton. By Proposition 5.1,

$$\begin{aligned}\widetilde{\mathcal{V}}(g) &= \int_{S^n} (4\pi)^{-n/2} \mathrm{e}^{-n/2} d\mu_{g_{S^n}} \\ &= \frac{\sqrt{2\pi} m^{m+\frac{1}{2}} \mathrm{e}^{-m}}{\Gamma(m+1)} \sqrt{\frac{2}{e}} \nearrow \sqrt{\frac{2}{e}} \quad \text{as } n \nearrow \infty.\end{aligned}$$

Here we have put $n = 2m+1, m \in \frac{1}{2}\mathbb{Z}^+$ and used that $\mathrm{Vol}(S^n, \frac{1}{2(n-1)} g_{S^n}) = 2\pi^{m+1}/\Gamma(m+1)$ and Stirling's formula:

$$\Gamma(m+1) = \sqrt{2\pi} m^{m+\frac{1}{2}} \mathrm{e}^{-m} \mathrm{e}^{\theta(m)} \text{ for } m > 0,$$

where $\theta(m) \searrow 0$ as $m \nearrow \infty$. This gives an upper bound for the constant ε_n obtained in Theorem 1.1:

$$\varepsilon_n \leq 1 - \mathrm{e}^{-\theta(m)} \sqrt{2\mathrm{e}^{-1}} \searrow 1 - \sqrt{2\mathrm{e}^{-1}} \text{ as } n \nearrow \infty.$$

Remark 6.3. Theorem 1.3 has another corollary which was pointed out by Professor Lei Ni.

Corollary 6.1. *Let $(M^n, g(t)), t \in [0, T)$ be a complete Ricci flow with bounded curvature and positive injectivity radius at $t = 0$ which develops singularity at finite time $t = T < \infty$. Then any singularity model of $(M^n, g(t))$ has finite fundamental group.*

The singularity model is the limit of dilations of $(M^n, g(t))$ around a singular point (see [10, Chapter 8] for the precise definition). We can take such a blow-up limit in the corollary by virtue of Perelman's no local collapsing theorem [25, Section 7] and Hamiton's compactness theorem [15]. The corollary immediately follows from the fact that such a singularity model is an ancient solution with positive asymptotic reduced volume (cf. [8, Lemma 8.22]).

We will be able to use this corollary in order to understand the singularities of the Ricci flow further. For example, we can prove the following: for any ancient solution $(N^{n-1}, g_N(t)), t \in (-\infty, \alpha)$, the canonical ancient solution on $S^1 \times N^{n-1}$ cannot occur as a blow-up limit of the Ricci flow as in Corollary 6.1. In the case where N is a sphere, this result was conjectured by Hamilton [14, Section 26] and proved by Ilmanen–Knopf [16].

Remark 6.4. (1) Feldman–Ilmanen–Ni [12] have discovered the *forward reduced volume* $\tilde{V}^+_{(p,0)}(t)$ for the (forward) Ricci flow $(M^n, g(t)), t \in [0, T)$ which

is non-increasing in t. However, its definition is given by

$$\tilde{V}^+_{(p,0)}(t) := \int_M (4\pi t)^{-n/2} \mathrm{e}^{\ell^+_{(p,0)}(\cdot,t)}\, d\mu_{g(t)}$$

(cf. with (5.6)) and it is not well defined for general non-compact manifolds. It is not likely that Theorem 1.1 has an analogue for the forward reduced volume $\tilde{V}^+_{(p,0)}(t)$.

(2) One can also easily generalize the monotonicity of $\tilde{V}^+_{(p,0)}(t)$ to the forward super Ricci flows $\frac{\partial}{\partial t} g \geq -2\,\mathrm{Ric}$, if the condition corresponding to Assumption 2.1 is imposed.

Remark 6.5. In Carrillo–Ni's preprint [4], the potential function f of the gradient Ricci soliton (M^n, g, f) is normalized so that

$$\int_M (4\pi\lambda)^{-n/2} \mathrm{e}^{-f}\, d\mu_g = 1.$$

Then their main result is the logarithmic Sobolev inequality for gradient Ricci solitons with $\mu(g, f) := \lambda(R + |\nabla f|^2) - f$ as the best constant. They also showed that $\mu(g, f) \geq 0$ for gradient shrinking Ricci solitons (under the curvature condition stronger than ours) and conjectured that $\mu(g, f) = 0$ implies that it is the Gaussian soliton. It is easily checked that $\mu(g, f) = -\log \mathrm{Vol}_f(M)$, where $\mathrm{Vol}_f(M)$ is the normalized f-volume of (M^n, g, f) with f being normalized in our sense as in (1.3). Hence, Corollary 1.1.(3) gives an affirmative answer to the conjecture in [4].

Remark 6.6. After the first version of this paper was completed, the result of Zhang [31] came to the author's attention. It states that for any gradient Ricci soliton (M, g, f), the completeness of g implies that of ∇f. Recall that we have used the assumption that $\mathrm{Ric} \geq -K$ for some $K \in \mathbb{R}$ in the proof of Corollary 1.1 only to ensure the completeness of ∇f and the existence of minimal $\mathcal{L}$-geodesics between any two points in space–time. A natural question is whether the assumption on Ric in the statement of Corollary 1.1 is superfluous.

Appendix

In this appendix, we present very detailed proofs to the facts on the super Ricci flow used in the proof of main theorem. The proofs rely on the following lemma whose proof in [25] works as well for the super Ricci flow.

Lemma A.1 ([25, Lemma 8.3]). *Let $(M^n, g(\tau))$ be a complete super Ricci flow.*

(a) *Assume that* $\mathrm{Ric}(\cdot, \tau_0) \le (n-1)K$ *on the ball* $B_{\tau_0}(x_0, r_0)$. *Then outside of* $B_{\tau_0}(x_0, r_0)$,

$$\Big(\frac{\partial}{\partial \tau} + \Delta_{g(\tau_0)}\Big) d_{\tau_0}(\cdot, x_0) \le (n-1)\Big(\frac{2}{3}Kr_0 + r_0^{-1}\Big).$$

The inequality is understood in the barrier sense.

(b) *Assume that* $\mathrm{Ric}(\cdot, \tau_0) \le (n-1)K$ *on the union of the balls* $B_{\tau_0}(x_0, r_0)$ *and* $B_{\tau_0}(x_1, r_0)$. *Then*

$$\frac{d^+}{d\tau} d_\tau(x_0, x_1)\Big|_{\tau=\tau_0} \le 2(n-1)\Big(\frac{2}{3}Kr_0 + r_0^{-1}\Big).$$

Here, $\frac{d^+}{d\tau} f(\tau) := \limsup_{\varepsilon \to 0+} \frac{f(\tau+\varepsilon)-f(\tau)}{\varepsilon}$ *denotes the upper Dini derivative.*

A.1 Perelman's Point picking lemma

Lemma A.2 ([25, Section 10; 17, Lemmas 30.1, 31.1]). *Let $(M^n, g(\tau)), \tau \in [0, T)$ be a complete super Ricci flow and $A, B > 0$ are arbitrary numbers. Assume that there exists a point $(x_1, \tau_1) \in M(B)$, where $M(B) := \{(x, \tau) \in M \times [0, T); |\mathrm{Rm}|(x, \tau)(T - \tau) > B\}$. Then we can find a point $(p_*, \tau_*) \in M(B)$ such that*

$$|\mathrm{Rm}|(x, \tau) \le 2|\mathrm{Rm}|(p_*, \tau_*) =: 2Q \tag{A.1}$$

for all (x, τ) with $d_{\tau_*}(x, p_*) < AQ^{-1/2}$ and $\tau_* \le \tau \le \tau_* + \frac{1}{2}BQ^{-1}$.

The proof is divided into two steps as in [25].

Claim 1. *Take $x_0 \in M$ and $A' > 0$ satisfying that $4(n-1)B\varepsilon \le 1$ and $(\varepsilon A')^2 \ge 3/2$ for some small $\varepsilon > 0$ and $A' \ge 2A$. Then we can find a point $(p_*, \tau_*) \in M(B)$ such that (A.1) holds for all (x, τ) with*

$$d_\tau(x, x_0) < d_{\tau_*}(p_*, x_0) + A'Q^{-1/2} \text{ and } \tau_* \le \tau \le \tau_* + \tfrac{1}{2}BQ^{-1}.$$

Proof. If not, we can construct a sequence $\{(x_i, \tau_i)\}_{i \in \mathbb{Z}^+} \subset M(B)$ starting from $(x_1, \tau_1) \in M(B)$ satisfying that

$$Q_{i+1} > 2Q_i, \quad d_{i+1} < d_i + A'Q_i^{-1/2} \quad \text{and} \quad \tau_i \le \tau_{i+1} \le \tau_i + \tfrac{1}{2}BQ_i^{-1}$$

where we put $Q_i := |\mathrm{Rm}|(x_i, \tau_i)$ and $d_i := d_{\tau_i}(x_i, x_0)$. We see that (x_{i+1}, τ_{i+1}) lies in $M(B)$ if (x_i, τ_i) does. Indeed,

$$\begin{aligned} Q_{i+1}(T-\tau_{i+1}) - B &> 2Q_i\Big(T-\tau_i-\tfrac{1}{2}BQ_i^{-1}\Big) - B \\ &= 2\Big(Q_i(T-\tau_i) - B\Big) > 0. \end{aligned}$$

This implies that $\{(x_i, \tau_i)\}_{i\in\mathbb{Z}^+} \subset M(B)$.

Then $Q_i > 2^{i-1}Q_1 \to \infty$ as $i \to \infty$, which contradicts to that

$$\tau_i \le \tau_1 + BQ_1^{-1} < T - \varepsilon_1 < T \quad \text{and} \quad d_i \le d_1 + 2A'Q_1^{-1/2}.$$

Here $\varepsilon_1 > 0$ is taken so that $|\mathrm{Rm}|(x_1, \tau_1)(T - \tau_1 - \varepsilon_1) > B$. Hence the sequence $\{(x_i, \tau_i)\}$ stops at finite steps and the terminal one is the desired point (p_*, τ_*). □

Claim 2. *The point (p_*, τ_*) just obtained satisfies the desired property.*

Proof. Take $x \in B_{\tau_*}(p_*, AQ^{-1/2})$ and put $r_0 := \varepsilon A'Q^{-1/2}$. Let $\tau' \in [\tau_*, \tau_* + \frac{1}{2}BQ^{-1}]$ be the supremum of τ'' such that

$$|\mathrm{Rm}|(\cdot, \tau) \le 2Q \text{ on } B_\tau(x_0, r_0) \cup B_\tau(x, r_0) \text{ for all } \tau \in [\tau_0, \tau''].$$

It follows easily from the choice of (p_*, τ_*) that $\tau' > \tau_*$ and $|\mathrm{Rm}| \le 2Q$ on $B_\tau(x_0, r_0)$ for $\tau \in [\tau_*, \tau_* + \frac{1}{2}BQ^{-1}]$.

Applying Lemma A.1(b) for $r_0 = \varepsilon A'Q^{-1/2}$,

$$\begin{aligned} d_{\tau'}(x, x_0) - d_{\tau_*}(x, x_0) &\le 2(n-1)\Big(\tfrac{4}{3}\varepsilon A'Q^{1/2} + (\varepsilon A')^{-1}Q^{1/2}\Big)(\tau' - \tau_*) \\ &\le 2(n-1)\varepsilon A'BQ^{-1/2} \\ &\le \tfrac{1}{2}A'Q^{-1/2}. \end{aligned}$$

Therefore, we have that

$$d_{\tau'}(x, x_0) \le d_{\tau_*}(x, p_*) + d_{\tau_*}(p_*, x_0) + \tfrac{1}{2}A'Q^{-1/2} < d_{\tau_*}(p_*, x_0) + A'Q^{-1/2}$$

and $\tau' = \tau_* + \frac{1}{2}BQ^{-1}$. As $x \in B_{\tau_*}(p_*, AQ^{-1/2})$ is arbitrary, we conclude that

$$|\mathrm{Rm}| \le 2Q \text{ on } B_{\tau_*}(p_*, AQ^{-1/2}) \times [\tau_*, \tau_* + \tfrac{1}{2}BQ^{-1}].$$

This completes the proof of the lemma. □

A.2 Ancient solutions have non-negative scalar curvature

Proposition A.1 ([6, Proposition 2.1])**.** *Any complete ancient super Ricci flow* $(M^n, g(\tau)), \tau \in [0,\infty)$ *satisfying (2.2) has non-negative trace of time derivative* $2H := \mathrm{tr}\frac{\partial}{\partial\tau}g \geq 0$.

Note that we have no assumption on the bound of $\frac{\partial}{\partial\tau}g$ in Proposition A.1.

Proof. We give a proof by contradiction which is based on the maximum principle argument. Assume that $H(x_0,\tau_0) < 0$ for some $(x_0,\tau_0) \in M\times[0,\infty)$. We may assume that $\tau_0 = 0$.

Let $\varphi : \mathbb{R} \to [0,1]$ is a non-increasing C^2 function satisfying that $\varphi = 1$ on $(-\infty, 1/2]$, $\varphi = 0$ on $[1,\infty)$ and $\varphi'' - \frac{2\varphi'^2}{\varphi} \geq -C\sqrt{\varphi}$ on $(-\infty,1)$ for some $C > 0$. Such a function can be constructed from $\varphi(s) = (s-1)^4$ for $s \leq 1$ near $s = 1$.

Take sufficiently large $T_0 > 0$ so that $n|H|(x_0,0)^{-1} \leq T_0$. Find $r_0 > 0$ such that $\mathrm{Ric} \leq (n-1)r_0^{-2}$ on $B_\tau(x_0,r_0)$ for all $\tau \in [0,T_0]$ and fix $A > 0$ so large enough that $|H|(x_0,0) \geq nC(Ar_0)^{-2}$.

Put

$$u(x,\tau) := \varphi\Big(\frac{d_\tau(x,x_0) - \frac{5}{3}(n-1)r_0^{-1}\tau}{Ar_0}\Big)H(x,\tau).$$

Then

$$\begin{aligned}\Big(\frac{\partial}{\partial\tau} + \Delta\Big)u(x,\tau) &= \varphi'\frac{(\frac{\partial}{\partial\tau} + \Delta)d_\tau(x,x_0) - \frac{5}{3}(n-1)r_0^{-1}}{Ar_0}H(x,\tau)\\ &\quad + \varphi\Big(\frac{\partial}{\partial\tau} + \Delta\Big)H(x,\tau) + \varphi''\frac{H(x,\tau)}{(Ar_0)^2} + 2\langle\nabla\varphi, \nabla H\rangle(x,\tau).\end{aligned}$$

Let $u_{\min}(\tau) := \min_{x\in M} u(x,\tau)$ and assume that $u_{\min}(\tau_1) = u(x_1,\tau_1) < 0$ for some $\tau_1 \geq 0$ and $x_1 \in M$. Then $d_{\tau_1}(x_1,x_0) < Ar_0 + \frac{5}{3}(n-1)r_0^{-1}\tau_1$ and $H(x_1,\tau_1) < 0$. Furthermore, we have $\nabla u(x_1,\tau_1) = 0$ and $\Delta u(x_1,\tau_1) \geq 0$.

If $d_{\tau_1}(x_1,x_0) < r_0$, then $u = H$ near (x_1,τ_1) and

$$\begin{aligned}\frac{d^+}{d\tau}u_{\min}(\tau_1) &\leq \liminf_{\tau\searrow\tau_1}\frac{u(x_1,\tau) - u(x_1,\tau_1)}{\tau - \tau_1}\\ &\leq -\Delta H(x_1,\tau_1) - 2|h|^2(x_1,\tau_1)\\ &\leq -\frac{2}{n}H^2(x_1,\tau_1) = -\frac{2}{n}u_{\min}(\tau_1)^2.\end{aligned}$$

If $d_{\tau_1}(x_1, x_0) \geq r_0$, by Lemma A.1(a) and that $2ab \leq a^2 + b^2$,

$$\begin{aligned}
\frac{d^+}{d\tau} u_{\min}(\tau_1) &\leq -2\varphi|h|^2(x_1,\tau_1) + \left(\varphi'' - \frac{2\varphi'^2}{\varphi}\right)\frac{H(x_1,\tau_1)}{(Ar_0)^2} \\
&\leq -\frac{2}{n}\varphi H^2(x_1,\tau_1) - C\sqrt{\varphi}\frac{H(x_1,\tau_1)}{(Ar_0)^2} \\
&\leq -\frac{2}{n}\varphi H^2(x_1,\tau_1) + \frac{nC^2}{2(Ar_0)^4} + \frac{1}{2n}\varphi H^2(x_1,\tau_1) \\
&= -\frac{1}{n}u_{\min}(\tau_1)^2 + \frac{nC^2}{2(Ar_0)^4} - \frac{1}{2n}u_{\min}(\tau_1)^2.
\end{aligned}$$

Since $|u_{\min}|(0) \geq |H|(x_0, 0) \geq nC(Ar_0)^{-2}$, we know that $\frac{d^+}{d\tau}u_{\min} \leq -\frac{1}{n}$ $u_{\min}^2$ on $[0, T_0]$. Therefore,

$$u_{\min}(\tau) \leq \frac{n}{nu_{\min}(0)^{-1} + \tau} \longrightarrow -\infty$$

as $\tau \to n|u_{\min}|(0)^{-1} \leq T_0$. This is the desired contradiction. $\square$

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